Studies on invariant rings and SAGBI bases

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博 士 論 文

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（不変式環とSAGBI基底の研究）

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Contents

Introduction

When a group acts on a polynomial ring, the invariant polynomials form a subalgebra. Then, it is a fundamental question to ask whether the invariant ring is finitely generated. It is also a problem to find an explicit generating set of the ring. These are always important problems in invariant theory. In this thesis, we also work on them.

The problem of finite generation of an invariant ring is closely related to the fourteenth problem of Hilbert which is stated as follows. Let k be a field, R a polynomial ring over k, and L the field of quotients of R. Assume that K is a subfield of L containing k. Then, is the k-subalgebra $K \cap R$ of R finitely generated? The first counterexample to this problem was constructed by Nagata in 1958. It was given as the invariant subring of a polynomial ring in 32 variables for a linear action of the 13-dimensional additive group. In 1990, P. Roberts gave a simple new counterexample of a different type. By following his methods, new counterexamples were constructed by some people. We also give new counterexamples of this type in Chapter 3.

One of the central theme of this thesis is derivations on a commutative algebra and their kernels. In the study of the fourteenth problem of Hilbert, they play an important role. For a commutative k-domain A, a k-linear map $D : A \to A$ is called a derivation on A if $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. Then, its kernel

$$A^D = \{a \in A \mid D(a) = 0\} \tag{1}$$

is a k-subalgebra of A. If D is moreover locally nilpotent, i.e., for each $a \in A$, there exists $r \in \mathbf{N}$ such that $D^r(a) = 0$, then A^D is equal to the invariant subring for an action of one dimensional additive group on A. The k-algebra A^D is not always finitely generated even if A is a polynomial ring. This is a kind of the fourteenth problem of Hilbert. We note that most of known counterexamples including those of Nagata and Roberts can be described as the kernels of derivations on polynomial rings.

Another underlying theme of this thesis is SAGBI bases, which are the natural Subalgebra Analogue to Gröbner Bases for Ideals. Let A be a k-subalgebra of the polynomial ring $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ in n variables over k, and $\preceq$ a monomial order on $k[\mathbf{x}]$. Then, we define the initial algebra $\mathrm{in}_{\preceq}(A)$ for $\preceq$ by the k-algebra generated by the set $\{\mathrm{in}_{\preceq}(f) \mid f \in A\}$ of initial terms. A generating set $\mathcal{S}$ of A over k is said to be a SAGBI basis for $\preceq$ if $\{\mathrm{in}_{\preceq}(f) \mid f \in \mathcal{S}\}$ generates $\mathrm{in}_{\preceq}(A)$. We call a subset $\mathcal{S}$ of A a universal SAGBI basis for A if it is a SAGBI basis for A with respect to any monomial order. The notion of SAGBI bases was introduced independently by Robbiano and Sweedler (1990) and Kapur and Madlener (1989). There exist indeed some applications of SAGBI bases to invariant theory and commutative algebra. However, compared with the theory of Gröbner bases, that of SAGBI bases has made a slow progress. One reason for this is the failure of some finiteness properties which Gröbner bases and initial algebras have. For example, the initial algebra of a finitely generated k-subalgebra is not always finitely generated, as we see in Chapter 4. In the theory of SAGBI bases, there are many problems remaining unsolved.

In the first three chapters, the characteristic of k is assumed to be zero. In Chapter 1, we study a kind of Zariski's problem which is a generalization of the fourteenth problem of Hilbert. Actually, we give a sufficient condition for finite generation of the kernel A^D of a derivation D. Assume that A is a finitely generated normal k-domain and the field K of quotients is a regular extension of k. Let Γ be an additive group, and assume that a Γ-grading is defined on A, i.e., $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ for some k-vector spaces $A_\gamma \subset A$ such that $A_\gamma A_\mu \subset A_{\gamma+\mu}$ for every $\gamma, \mu \in \Gamma$. Assume further that D is Γ-homogeneous, i.e., there exists $\gamma_0 \in \Gamma$ such that $D(A_\gamma) \subset A_{\gamma+\gamma_0}$ for every $\gamma \in \Gamma$. We denote the unique extension of D to K by the same symbol D, and define the subfield L of K^D by

$$L = \{f/g \mid f, g \in A_\gamma \text{ for some } \gamma \in \Gamma \text{ with } g \neq 0 \text{ and } D(f/g) = 0\}. \tag{2}$$

Then, we prove that A^D is finitely generated over k if the filed L is a simple extension of k (Theorem 1.2).

Assume that D is a derivation on $k[\mathbf{x}]$. We define the support $\mathrm{supp}(D)$ of D to be the set of $(a_1, \ldots, a_n) \in \mathbf{Z}^n$ such that the monomial $x_1^{a_1} \cdots x_n^{a_n}$ appears in $x_i^{-1}D(x_i)$ with nonzero coefficient for some i. One consequence of Theorem 1.2 is that the kernel $k[\mathbf{x}]^D$ is finitely generated over k if the dimension of $\mathrm{supp}(D)$ is at most two. Here, we define the dimension of a subset $S \subset \mathbf{R}^n$ to be the dimension of the convex hull of S in $\mathbf{R}^n$ if $S \neq \emptyset$, and -1 if $S = \emptyset$. We note that there exists a derivation on $k[\mathbf{x}]$ whose kernel is not finitely generated if the dimension of its support is more than two. By using this criterion, we show the finite generation of the kernels of several derivations on $k[\mathbf{x}]$. For example, in case of $n = 4$, it was an open question whether the kernel $k[\mathbf{x}]^D$ of a locally nilpotent derivation D on $k[\mathbf{x}]$ is finitely generated even if each $D(x_i)$ is a monomial multiplied by a scalar. However, by the criterion, we conclude that such $k[\mathbf{x}]^D$ is always finitely generated if each $D(x_i)$ is a monomial multiplied by a scalar.

Chapter 2 deals with a derivation on a polynomial ring, and gives a sufficient condition for its kernel to have a finite universal SAGBI basis. First, we define a subset $\mathrm{supp}^\circ(D)$ of $\mathrm{supp}(D)$ for a derivation D on $k[\mathbf{x}]$. It is an improvement of the notion of the support. The result is that the kernel $k[\mathbf{x}]^D$ has a finite universal SAGBI basis if the dimension of $\mathrm{supp}^\circ(D)$ is at most two. This criterion is a generalization of that in the previous chapter. A derivation D on $k[\mathbf{x}]$ is said to be triangular if each $D(x_i)$ is contained in $k[x_1, \ldots, x_{i-1}]$. In case that D is triangular, we get a stronger statement that there exists a universal SAGBI basis for $k[\mathbf{x}]^D$ with at most n elements if the dimension of $\mathrm{supp}^\circ(D)$ is at most two. Furthermore, we describe the universal SAGBI basis explicitly. This contains the results of Maubach (2000) and Khoury (2001) about the kernels of some triangular derivations as special cases.

In Chapter 3, we generalize Roberts' counterexample to the fourteenth problem of Hilbert. Our result shows that there exist a lot of counterexamples of this type. Let $k[\mathbf{x}][\mathbf{y}] = k[x_1, \ldots, x_m][y_1, \ldots, y_n]$ be the polynomial ring in $m + n$ variables over k. In case of $n = m + 1$, the derivation $D_{t,m}$ on $k[\mathbf{x}][\mathbf{y}]$ is defined by $D_{t,m}(x_i) = 0$ and

$D_{t,m}(y_i) = x_i^{t+1}$ for $i = 1, \ldots, m$ and $D_{t,m}(y_{m+1}) = (x_1 \cdots x_m)^t$. Deveney and Finston (1992) showed that the counterexample of Roberts is obtained as the kernel $k[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of $D_{t,m}$ for $m = 3$ and $t \geq 2$. Kojima and Miyanishi (1997) showed further that $k[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ is not finitely generated over k if $m \geq 3$ and $t \geq 2$. We consider, more generally, a derivation D on $k[\mathbf{x}][\mathbf{y}]$ for $n \geq 4$ and $m \geq n-1$ of the form $D(x_i) = 0$ for all i and $D(y_j) = x_1^{\delta_j^1} \cdots x_m^{\delta_j^m}$ for each j, where $\delta_j^l \in \mathbf{Z}_{\geq 0}$. Our result is the following. For each i, j, l, we set $\epsilon_{i,j}^l = \delta_i^l - \delta_j^l$. Assume that $\epsilon_{i,j}^i > 0$ for any $1 \leq i \leq n-1$ and $1 \leq j \leq n$ with $i \neq j$. We define

$$\eta = \frac{\epsilon_{1,n}^1}{\min\{\epsilon_{1,j}^1 \mid j = 2, \ldots, n-1\}}, \tag{3}$$

and

$$\eta_{l,i} = \eta \min\{\max\{\epsilon_{1,l}^i, \epsilon_{2,l}^i\}, 0\} \tag{4}$$

for $i = 2, \ldots, n-1$ and $l = 3, \ldots, n-1$. Then, our theorem asserts that $k[\mathbf{x}][\mathbf{y}]^D$ is not finitely generated over k if the system of linear inequalities

$$\begin{cases} u_1 + \cdots + u_{n-2} = 1 \\ u_1 \geq \eta,\ u_i \geq 0 \ (i = 2, \ldots, n-2) \\ \displaystyle\sum_{j=1}^{n-2} \min\{\epsilon_{n,1}^i, \epsilon_{n,j+1}^i\} u_j + \eta_{l,i} \geq 0 \ (i = 2, \ldots, n-1) \end{cases} \tag{5}$$

in the $n-2$ variables $u_1, \ldots, u_{n-2}$ has a solution in $\mathbf{R}^{n-2}$ for each $l = 3, \ldots, n-1$. If $n = 4$, then the condition that (5) has a solution for $l = 3$ is equivalent to

$$\frac{\epsilon_{1,4}^1}{\min\{\epsilon_{1,2}^1, \epsilon_{1,3}^1\}} + \frac{\epsilon_{2,4}^2}{\min\{\epsilon_{2,3}^2, \epsilon_{2,1}^2\}} + \frac{\epsilon_{3,4}^3}{\min\{\epsilon_{3,1}^3, \epsilon_{3,2}^3\}} \leq 1. \tag{6}$$

In case $(m, n) = (3, 4)$, for instance, there exist 2450001 derivations on $k[\mathbf{x}][\mathbf{y}]$ which satisfy (6) and $\gcd\{\mathbf{x}^{\delta_1}, \mathbf{x}^{\delta_2}, \mathbf{x}^{\delta_3}, \mathbf{x}^{\delta_4}\} = 1$ even if we impose the restriction $\delta_i^l \leq 10$ for all i, l. We deduce from this criterion that $k[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ is not finitely generated if $t = 1, m \geq 4$ besides the case where $t \geq 2, m \geq 3$.

In this chapter, we also study Roberts' original counterexample further, and determine its initial algebra explicitly. Consequently, it turn out that an infinite system of invariants appeared in Roberts' proof is actually a SAGBI basis for his k-algebra. We also give some sufficient condition for finite generation.

In Chapter 4, we discuss SAGBI bases for the invariant subrings of a polynomial ring for certain action of finite groups. It is well-known that the invariant subring $k[\mathbf{x}]^{S_n}$ of $k[\mathbf{x}]$ for the symmetric group S_n acting by permutations of the variables is generated by elementary symmetric polynomials in $x_1, \dots, x_n$ over k. Actually, they form a universal SAGBI basis for $k[\mathbf{x}]^{S_n}$. Assume that G is a subgroup of S_n. Then, the invariant subring $k[\mathbf{x}]^G$ for G is defined by restricting the action of S_n to G. We show that, for any monomial order $\preceq$ on $k[\mathbf{x}]$, the initial algebra $\mathrm{in}_{\preceq}(k[\mathbf{x}]^G)$ is finitely generated if and only if G is generated by transpositions. Furthermore, for each G, the set of initial algebras $\mathrm{in}_{\preceq}(k[\mathbf{x}]^G)$ for all monomial orders $\preceq$ on $k[\mathbf{x}]$ is equal to the order of G if G is generated by transpositions, while it is uncountable otherwise. This result contains that of Göbel (1998) on the condition for finite generation of $\mathrm{in}_{\preceq_{\mathrm{lex}}}(k[\mathbf{x}]^G)$ for a lexicographic order $\preceq_{\mathrm{lex}}$. It is easy to see the result if G is generated by transpositions. In the proof of the other case, we employed a topological method as follows. We introduce a structure of a compact metric space on the set of monomial orders on $k[\mathbf{x}]$. If $\mathrm{in}_{\preceq}(A)$ is finitely generated for a k-subalgebra A of $k[\mathbf{x}]$ and a monomial order $\preceq$, then the set of monomial orders $\preceq'$ with $\mathrm{in}_{\preceq'}(A) = \mathrm{in}_{\preceq}(A)$ is open with respect to this topology. We show that each set of monomial orders which define the same initial algebra of $k[\mathbf{x}]^G$ is nowhere dense if G is not generated by transpositions. This implies that the initial algebras are not finitely generated. Furthermore, there are uncountable distinct initial algebras for each $k[\mathbf{x}]^G$ by Baire's lemma. We also get a similar result for the invariant subrings of a Laurent polynomial ring.

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Chapter 1

A condition for finite generation of the kernel of a derivation

A condition for finite generation of the kernel of a derivation

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Abstract

We consider a homogeneous derivation on a finitely generated graded normal domain over a field, and give a sufficient condition for finite generation of its kernel. As its consequence, we show some sufficient conditions that the kernel of a derivation on a polynomial ring is finitely generated. In particular, we prove that the kernel of a locally nilpotent monomial derivation on a polynomial ring in four variables is finitely generated.

Key words: derivations, the fourteenth problem of Hilbert.

1 Introduction

It is an important problem to find a criterion for finite generation of the kernel of a derivation on a polynomial ring. It is closely related to the fourteenth problem of Hilbert which is stated as follows. Let k be a field and $x_1, \ldots, x_n$ algebraically independent elements over k. Assume that L is a subfield of $k(x_1, \ldots, x_n)$ containing k. Then, is the k-algebra $L \cap k[x_1, \ldots, x_n]$ finitely generated? In [16], Zariski generalized this problem

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as follows. Let A be a finitely generated normal domain over k, K its fractional field. Then, is the k-algebra $A \cap L$ finitely generated for a subfield $L \subset K$ containing k? A counterexample to Zariski's problem was first found by Rees [14]. Then, Nagata gave the first counterexample to Hilbert's problem [11]. Recently, Daigle and Freudenburg gave a new counterexample as the kernel of a derivation on a polynomial ring in five variables [1] (See also [3, 8, 15]). We are interested to find a sufficient condition for finite generation.

Zariski also showed in the same paper that the answer to his problem is affirmative if the transcendence degree of L over k is at most two. By using his result, Nagata and Nowicki showed that the kernel of a derivation on a polynomial ring in at most three variables is finitely generated if the characteristic of k is zero [12]. In the present paper, we will give another sufficient condition for the finite generation.

Throughout this paper, we will denote by k a field of characteristic zero. For a commutative k-algebra A, a k-linear map $D : A \to A$ is called a k-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. For a k-vector subspace $V \subset A$, we will denote by

$$V^D = \{f \in V \mid D(f) = 0\}. \tag{1.1}$$

Note that, if V is a k-subalgebra of A, then V^D is a k-subalgebra of V.

Let A be a finitely generated normal domain over k, K the fractional field of A, and D a k-derivation on A. Then, D is extended uniquely to a k-derivation on K. We denote it by the same symbol D. We will consider the sufficient condition that the k-algebra A^D is finitely generated. We note that Zariski's theorem mentioned above implies the following.

Theorem 1.1 (Zariski [16]) *If* $\operatorname{trans.deg}_k A^D \leq 2$*, then* A^D *is finitely generated over* k.

Here, we denote by $\operatorname{trans.deg}_k R$ the transcendence degree of R over k for an integral domain R containing k.

Now, let Γ be an additive group, and assume that a Γ-grading is defined on A, that is, $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ for some k-vector spaces $A_\gamma \subset A$ such that $A_\gamma A_{\gamma'} \subset A_{\gamma+\gamma'}$ for every $\gamma, \gamma' \in \Gamma$. Since A is a domain, the set $A_H = \bigcup_{\gamma\in\Gamma} A_\gamma \setminus \{0\}$ is multiplicatively closed. We set $B = A_H^{-1}A$ the localization of A by A_H. Then, the Γ-grading $B = \bigoplus_{\gamma\in\Gamma} B_\gamma$ is defined by setting

$$B_\gamma = \{f/g \mid (f, g) \in A_{\gamma'+\gamma} \times (A_{\gamma'} \setminus \{0\}) \text{ for some } \gamma' \in \Gamma\} \tag{1.2}$$

for each $\gamma \in \Gamma$. Note that B_0 is a field containing k. We say that D is Γ-homogeneous if there exists $\gamma_0 \in \Gamma$ such that $D(A_\gamma) \subset A_{\gamma+\gamma_0}$ for every $\gamma \in \Gamma$. If this is the case, then it follows that

$$A^D = \bigoplus_{\gamma\in\Gamma} A_\gamma^D. \tag{1.3}$$

Of course, if $\Gamma = \{0\}$, then every k-derivation on A is Γ-homogeneous.

Before stating our result, we assume that K/k is a regular extension by the following reason. Set $k' = \bar{k} \cap K$, where $\bar{k}$ is the algebraic closure of k. Then, K/k' is a regular extension, since k is of characteristic zero. Since k' is separable algebraic over k, we have $k' \subset K^D$ (cf. [9, Chapter X, Proposition 7]). So, we may regard D as a k'-derivation on A. Since k'/k is a finite extension, finite generation of A^D over k' implies that over k. Thus, by replacing k by k', we may assume that K is a regular extension of k.

Now, here is our main theorem.

Theorem 1.2 *Let A be a finitely generated normal domain over k, let K be the fractional field of A, and let D be a k-derivation on A. Assume that K/k is a regular extension, and D is Γ-homogeneous for some additive group Γ. If B_0^D/k is a simple extension, then A^D is finitely generated over k. In particular, if A is rational, then* $\text{trans.deg}_k B_0^D \leq 1$ *implies finite generation of A^D over k.*

Here, we say that A is rational if the fractional field of A is a rational function field over k. Since K/k is a regular extension, the condition that B_0^D/k is a simple extension implies that $B_0^D = k$ or B_0^D is a rational function field of one variable over k.

By applying Theorem 1.2 to a polynomial ring, we get the following theorem. Let $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, and D a nonzero k-derivation on $k[\mathbf{x}]$. Then, let $\{\delta_1, \ldots, \delta_m\} \subset \mathbf{Z}^n$ be the minimal subset such that the Laurent polynomial $x_j^{-1}D(x_j)$ is written as $\sum_{i=1}^m \kappa_{i,j}\mathbf{x}^{\delta_i}$ with $\kappa_{i,j} \in k$ for every j. Here, for each $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n$, we denote by $\mathbf{x}^a$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$.

Theorem 1.3 *Assume that $\delta_1, \ldots, \delta_m$ lie on a two dimensional affine subspace of $\mathbf{R}^n$. Then, $k[\mathbf{x}]^D$ is finitely generated.*

Here, we note that the assumption of Theorem 1.3 is equivalent to the assumption that the dimension of the vector subspace of $\mathbf{R}^n$ generated by $\{\delta_i - \delta_j \mid 1 \leq i, j \leq n\}$ is at most two.

One may notice a similarity between Theorems 1.1 and 1.2. However, in the proof of Theorem 1.2, we use Gordan's lemma rather than Zariski's theorem, to show the finite generation. We will prove Theorem 1.2 in Section 2. In Section 3, we prove Theorem 1.3. As one of the corollaries, we show that the kernel of a locally nilpotent monomial derivation on a polynomial ring in four variables is finitely generated. This result is new.

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2 The proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Let L be an arbitrary subfield of K containing k, and $g_1, \ldots, g_r$ be elements of $K \setminus \{0\}$. Then, consider the k-subalgebra

$$R = \sum_{i_1, \ldots, i_r \in \mathbf{Z}} \left(Lg_1^{i_1} \cdots g_r^{i_r} \cap A\right) \tag{2.1}$$

of A.

Lemma 2.1 *Assume that K/k is a regular extension, and L/k is a simple extension. Then, R is finitely generated over k.*

Before proving Lemma 2.1, we will show that Theorem 1.2 is a consequence of this lemma. Since A is finitely generated over k, we may assume that Γ is finitely generated, by replacing Γ by the submodule generated by $\{\gamma \mid A_\gamma \neq \{0\}\}$ if necessary.

Assume that there exists $0 \neq g \in B_\gamma^D$ for $\gamma \in \Gamma$. Then, we have

$$A_\gamma^D = B_0^D g \cap A. \tag{2.2}$$

Actually, if $g' \in A_\gamma^D$, then $g'/g \in B_0^D$. Hence, $g' \in B_0^D g \cap A$. The converse is readily verified. Thus, we have (2.2).

Now, we set

$$\Gamma_0 = \left\{\gamma \in \Gamma \mid B_\gamma^D \neq \{0\}\right\}. \tag{2.3}$$

Then, Γ_0 is a subgroup of Γ. Since Γ is finitely generated by assumption, Γ_0 is finitely generated. Let $\gamma_1, \ldots, \gamma_r$ be generators of Γ_0, and take $0 \neq g_i \in B_{\gamma_i}^D$ for each $1 \leq i \leq r$. Then, we have

$$A^D = \sum_{i_1,\ldots,i_r \in \mathbf{Z}} \left(B_0^D g_1^{i_1} \cdots g_r^{i_r} \cap A\right) \tag{2.4}$$

by (1.3) and (2.2). By assumption, B_0^D/k is a simple extension. Hence, (2.4) is finitely generated by Lemma 2.1.

The last assertion follows immediately from the first part if $\operatorname{trans.deg}_k B_0^D = 0$. If $\operatorname{trans.deg}_k B_0^D = 1$, then we are done since B_0^D is a rational function field over k by Lüroth's theorem.

In the rest of the section, we will prove Lemma 2.1.

Lemma 2.2 *Assume that k is algebraically closed. Let Σ be a set of closed points on $\mathbf{P}^1$ with at least two elements. Then, for any divisor $E = \sum_{p \in \Sigma} m_p p$ on $\mathbf{P}^1$, the k-vector space $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(E))$ is generated by elements whose supports of zeros and poles are contained in Σ.*

Proof. Let $k[y_0, y_1]$ be the homogeneous coordinate ring of $\mathbf{P}^1$, and assume that each $p \in \mathbf{P}^1$ corresponds to the homogeneous prime ideal $(\alpha_p y_0 - \beta_p y_1) \subset k[y_0, y_1]$ for $\alpha_p, \beta_p \in k$. By a projective transformation, we may assume that the two points corresponding to the homogeneous prime ideals (y_0) and (y_1) are contained in Σ.

Assume that $0 \neq f \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(E))$. Then, it follows that

$$f = \alpha \prod_{p \in \mathbf{P}^1} (\alpha_p y_0 - \beta_p y_1)^{n_p} \prod_{p \in \Sigma} (\alpha_p y_0 - \beta_p y_1)^{-m_p} \tag{2.5}$$

for $n_p \geq 0$ with $\sum_{p \in \mathbf{P}^1} n_p = \deg E$, and some $\alpha \in k^\times$. Hence, f is a linear combination of

$$y_0^i y_1^{\deg E - i} \prod_{p \in \Sigma} (\alpha_p y_0 - \beta_p y_1)^{-m_p} \tag{2.6}$$

for $0 \leq i \leq \deg E$. The supports of zeros and poles of (2.6) are contained in Σ. Therefore, the lemma is proved. $\square$

The proof of Lemma 2.1. First, we show that it suffices to prove the lemma when k is algebraically closed. Set $\bar{L} = L \otimes_k \bar{k}$ and $\bar{A} = A \otimes_k \bar{k}$. As the flat base change of (2.1), we have

$$R \otimes_k \bar{k} = \sum_{i_1, \ldots, i_r \in \mathbf{Z}} \left(\bar{L}(g_1^{i_1} \cdots g_r^{i_r} \otimes 1) \cap \bar{A} \right). \tag{2.7}$$

Since $\bar{k}/k$ is a field extension, finite generation of the $\bar{k}$-algebra $R \otimes_k \bar{k}$ implies that of R. Therefore, we may assume that k is algebraically closed.

Assume first that L is a rational function field of one variable over k. Then, the dominant rational map

$$\phi : \operatorname{Spec} A \dashrightarrow \mathbf{P}^1 \tag{2.8}$$

is defined by the inclusion map $\iota : L \to K$. Since A is normal and $\mathbf{P}^1$ is complete, ϕ is defined at the generic point of every prime divisor of $\operatorname{Spec} A$. Consider the homomorphism

$$\phi^* : \operatorname{Div}(\mathbf{P}^1) \to \operatorname{Div}(\operatorname{Spec} A) \tag{2.9}$$

of the divisor groups of $\mathbf{P}^1$ and $\operatorname{Spec} A$. Clearly, we have $\phi^*((f)) = (\iota(f))$ for $f \in L^\times$. Since ϕ is dominant, the complement of the image is a finite set. Hence, $\ker \phi^*$ is finitely generated.

We may find a finite subset $\Sigma \subset \mathbf{P}^1$ of closed points having at least two elements with the following properties:

(i) $\ker \phi^*$ is contained in the subgroup of $\operatorname{Div}(\mathbf{P}^1)$ generated by Σ, where we regard Σ as a set of prime divisors.

(ii) Let p be the generic point of a prime divisor appearing in $(g_i) \in \operatorname{Div}(\operatorname{Spec} A)$ for some $1 \leq i \leq r$. Then, $\phi(p)$ is in Σ, unless it is the generic point of $\mathbf{P}^1$.

Let S be the set of $((a_i)_{i=1}^r, (b_p)_{p\in\Sigma}) \in \mathbf{Z}^r \times \mathbf{Z}^\Sigma$ such that $\sum_{i=1}^r a_i(g_i) + \sum_{p\in\Sigma} b_p\phi^*(p)$ is effective and $\sum_{p\in\Sigma} b_p = 0$. Then, S is a finitely generated subsemigroup of $\mathbf{Z}^r \times \mathbf{Z}^\Sigma$ by Gordan's lemma [13, Proposition 1.1.(ii)]. Assume that $S' \subset S$ is a finite subset of generators. For each $s = ((a_i)_{i=1}^r, (b_p)_{p\in\Sigma}) \in S'$, we take $h \in L^\times$ such that $(h) = \sum_{p\in\Sigma} b_p p$, and set $f_s = hg_1^{a_1} \cdots g_r^{a_r}$. Then, f_s is uniquely determined up to multiplications of elements in $k^\times$ by $s \in S'$. By definition, it follows that $f_s \in R$. We will show that $R = R'$ for $R' = k[\{f_s \mid s \in S'\}]$.

Let $f = hg_1^{a_1} \cdots g_r^{a_r} \in A$ for $h \in L^\times$ and $(a_i)_i \in \mathbf{Z}^r$. Set $(h) = \sum_p b_p p$, and put $E = \sum_{p\in\Sigma} b_p p$. If $(h) = E$, then $((a_i)_{i=1}^r, (b_p)_{p\in\Sigma}) \in S$. Write $((a_i)_{i=1}^r, (b_p)_{p\in\Sigma}) = \sum_{s\in S'} c_s s$ for $c_s \in \mathbf{Z}_{\geq 0}$. Then, f is equal to $\prod_{s\in S'} f_s^{c_s}$ multiplied by an element in $k^\times$. Hence, $f \in R'$. For the general case, we remark that $h \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-E))$, and $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-E))g_1^{a_1} \cdots g_r^{a_r} \subset A$. By Lemma 2.2, h is a linear combination of $h' \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-E))$ whose supports of zeros and poles are contained in Σ. However, for such h', we already have $h'g_1^{a_1} \cdots g_r^{a_r} \in R'$ as above. Hence, $f \in R'$. This proves that $R = R'$.

Now, assume that $L = k$. Then, consider the similar S as above, but $\Sigma = \emptyset$. We define S' and R' similarly. Then, it is easy to see that $R = R'$. Therefore, the proof of Lemma 2.1 is completed. $\square$

3 Applications

In this section, we apply Theorem 1.2 to a derivation on a polynomial ring.

Let $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, $k(\mathbf{x})$ the field of fractions of $k[\mathbf{x}]$, and D a nonzero k-derivation on $k[\mathbf{x}]$. We also denote by D the unique extension of D to the k-derivation on $k(\mathbf{x})$. It is well-known that the set of k-derivations on $k[\mathbf{x}]$ is the free $k[\mathbf{x}]$-module with basis $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$. So, D can be written as

$$D = D(x_1)\frac{\partial}{\partial x_1} + \cdots + D(x_n)\frac{\partial}{\partial x_n}. \tag{3.1}$$

Recall that $\{\delta_1, \ldots, \delta_m\} \subset \mathbf{Z}^n$ is the minimal subset such that $x_j^{-1}D(x_j)$ is written as $\sum_{i=1}^m \kappa_{i,j}\mathbf{x}^{\delta_i}$ with $\kappa_{i,j} \in k$ for every j. Then, we may express (3.1) as $D = \sum_{i=1}^m D_i$, where

$$D_i = \mathbf{x}^{\delta_i}\left(\kappa_{i,1}x_1\frac{\partial}{\partial x_1} + \cdots + \kappa_{i,n}x_n\frac{\partial}{\partial x_n}\right). \tag{3.2}$$

Set $M = \sum_{i=1}^m \mathbf{Z}(\delta_i - \delta_1)$, and $k(M) = k(\{\mathbf{x}^{\delta_i}/\mathbf{x}^{\delta_1} \mid i = 1, \ldots, m\})$. Then, $\operatorname{trans.deg}_k k(M)$ is equal to the rank of the $\mathbf{Z}$-module M. We define the k-derivation D' on $k(\mathbf{x})$ by $D'(f) = \mathbf{x}^{-\delta_1}D(f)$ for $f \in k(\mathbf{x})$. It induces a k-derivation on $k(M)$. We remark that $V^{D'} = V^D$ for any k-vector subspace $V \subset k(\mathbf{x})$. Actually, $\mathbf{x}^{-\delta_1}D(f) = 0$ if and only if $D(f) = 0$ for $f \in k(\mathbf{x})$.

Now, let $\Gamma = \mathbf{Z}^n/M$, and define the Γ-grading $k[\mathbf{x}] = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma$ by setting $k[\mathbf{x}]_\gamma$ the k-vector space generated by monomials $\mathbf{x}^a \in k[\mathbf{x}]$ with $\bar{a} = \gamma$ for each $\gamma \in \Gamma$. Here, we denote by $\bar{a}$ the image of a in $\mathbf{Z}^n/M$ for each $a \in \mathbf{Z}^n$. Then, D is Γ-homogeneous with respect to this Γ-grading. As in Section 1, let B denote the localization of $k[\mathbf{x}]$ by $\bigcup_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma \setminus \{0\}$, and define B_γ similarly to (1.2) for each $\gamma \in \Gamma$. Then, it follows that $B_0 = k(M)$. Therefore, by Theorem 1.2, we have the following.

Proposition 3.1 *If* $\operatorname{trans.deg}_k k(M)^D \leq 1$, *then* $k[\mathbf{x}]^D$ *is finitely generated.*

In order to prove Theorem 1.3, we need the following lemma.

Lemma 3.2 *Assume that* $k(M)^{D'} = k(M)$. *Then,* $k[\mathbf{x}]^D$ *is finitely generated.*

Proof. The assumption implies that

$$k[\mathbf{x}]^D = \bigoplus_{\gamma \in \Gamma_0} k[\mathbf{x}]_\gamma. \tag{3.3}$$

Here, we define $\Gamma_0 \subset \Gamma$ as (2.3). Hence, we have $k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in S\}]$, where

$$S = \{a \in \mathbf{Z}^n \mid \bar{a} \in \Gamma_0\} \cap \mathbf{Z}_{\geq 0}{}^n. \tag{3.4}$$

By Gordan's lemma [13, Proposition 1.1.(ii)], S is a finitely generated subsemigroup of $\mathbf{Z}_{\geq 0}{}^n$. Therefore, $k[\mathbf{x}]^D$ is finitely generated. □

Now, we prove Theorem 1.3. The condition implies that the rank of M is at most two. Hence, $\text{trans.deg}_k\, k(M) \leq 2$. Assume that $\text{trans.deg}_k\, k(M)^{D'} = 2$. Then, $k(M)/k(M)^{D'}$ is a separable algebraic extension. Hence, D' is zero on $k(M)$ (cf. [9, Chapter X, Proposition 7]), that is, $k(M)^{D'} = k(M)$. Hence, $k[\mathbf{x}]^D$ is finitely generated by Lemma 3.2. If $\text{trans.deg}_k\, k(M)^{D'} \leq 1$, then we are done by Proposition 3.1. Actually, $k(M)^{D'} = k(M)^D$, as we remarked above. Therefore, Theorem 1.3 is proved.

The k-derivation D on $k[\mathbf{x}]$ is said to be *locally nilpotent* if, for each $f \in k[\mathbf{x}]$, there exists $l \geq 0$ such that $D^l(f) = 0$. It is said to be *triangular* if $D(x_i) \in k[x_1, \ldots, x_{i-1}]$ for every i. Note that, if D is triangular, then it is locally nilpotent. We say D is *monomial* if each $D(x_i)$ is a monomial multiplied by an element in k.

Corollary 3.3 *Assume that $n \geq 3$ and D is a monomial derivation on $k[\mathbf{x}]$. If $D(x_i) = 0$ for $1 \leq i \leq n-3$, then $k[\mathbf{x}]^D$ is finitely generated.*

Proof. The condition implies $m \leq 3$. Hence, the corollary follows from Theorem 1.3. □

For $n = 5$, the triangular monomial k-derivation

$$D = x_1^3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5} \tag{3.5}$$

was studied by Daigle and Freudenburg [1]. For this derivation, $k[\mathbf{x}]^D$ is not finitely generated. Namely, it is a counterexample to the fourteenth problem of Hilbert. Note that, in this case, the dimension of the affine subspace of $\mathbf{R}^n$ generated by $\{\delta_1, \ldots, \delta_m\}$ is three.

For $n = 4$, we have the following.

Corollary 3.4 *Assume that $n = 4$. If D is triangular and monomial, then $k[\mathbf{x}]^D$ is finitely generated.*

Proof. Assume that $D(x_1) \neq 0$. Then, we have $D(x_1) \in k \setminus \{0\}$, since D is triangular. Hence, D has a slice. Here, we say an element $s \in k[\mathbf{x}]$ is a *slice* of D if $D(s) = 1$. It is known that the kernel of a locally nilpotent k-derivation on $k[\mathbf{x}]$ with a slice is finitely generated (cf. [5, Corollary 1.3.23]). Hence, $k[\mathbf{x}]^D$ is finitely generated. Assume that $D(x_1) = 0$. Then, $k[\mathbf{x}]^D$ is finitely generated by Corollary 3.3. $\square$

Actually, Maubach showed that, under the assumption in Corollary 3.4, $k[\mathbf{x}]^D$ is generated by at most four elements [10]. Moreover, Daigle and Freudenburg showed in [2] that the kernel of any triangular derivation is finitely generated if $n = 4$, though there is no general bound for the minimal number of generators.

Since D is locally nilpotent if it is triangular, the following corollary is stronger than Corollary 3.4. The possibility of this corollary was suggested by the referee.

Corollary 3.5 *Assume that $n = 4$. If D is locally nilpotent and monomial, then $k[\mathbf{x}]^D$ is finitely generated.*

Proof. We set $x_i^{-1}D(x_i) = \kappa_i \mathbf{x}^{\delta_i}$ with $\kappa_i \in k$ for each i, and denote by $\delta_{i,j}$ the j-th component of δ_i. Note that $\delta_{i,j} \geq 0$ for $i \neq j$, and $\delta_{i,i} \geq -1$ for any i. We show that, if D is not triangular for any change of indices, then the set $\{\delta_i \mid \kappa_i \neq 0\}$ is contained in some two dimensional affine subspace of $\mathbf{R}^4$. If it is proved, then the assertion follows from Corollary 3.4 and Theorem 1.3.

Suppose the contrary, that is, $\kappa_i \neq 0$ for all i and the set $\{\delta_1, \ldots, \delta_4\}$ was not contained in any two dimensional affine subspace of $\mathbf{R}^4$. Since D is not triangular for any change of indices by assumption, there exists $1 \leq p \leq 4$ and distinct $1 \leq i_1, \ldots, i_p \leq 4$ such that

$\delta_{i_1,i_2}, \delta_{i_2,i_3}, \ldots, \delta_{i_p,i_1} > 0$ if $p > 1$, and $\delta_{i_1,i_1} \geq 0$ if $p = 1$. Actually, if such p and $i_1, \ldots, i_p$ do not exist, then we may change indices so that $\delta_{i,j} > 0$ implies $j < i$ for any distinct $1 \leq i, j \leq 4$. This implies that D is triangular.

We define a sequence $\{u_j\}_{j=1}^{\infty}$ by $u_j = i_q$, where $1 \leq q \leq p$ is the integer with $q \equiv j \pmod{p}$. Then, it follows that

$$\prod_{q=0}^{l-1}(1 + \sum_{j=1}^{q} \delta_{u_j,u_{q+1}}) > 0 \tag{3.6}$$

for any $l > 0$. Actually, if $p = 1$, then $u_j = i_1$ for all j, and $\delta_{i_1,i_1} \geq 0$. Hence, $1 + \sum_{j=1}^{q} \delta_{u_j,u_{q+1}} > 0$ for each q. Assume that $p > 1$. Let $q > 0$, and take $q', q'' \in \mathbf{Z}$ so that $q = q'p + q''$ and $0 \leq q'' < p$. If $\delta_{u_j,u_{q+1}} < 0$, then $u_j = u_{q+1}$ and $\delta_{u_j,u_{q+1}} = -1$. The number of such indices $1 \leq j \leq q$ is equal to q'. However, the number of such indices $1 \leq j \leq q$ that $u_j = u_{q+1} - 1$ is greater than or equal to q'. For such j, we have $\delta_{u_j,u_{q+1}} \geq 1$. Hence, we have $1 + \sum_{j=1}^{q} \delta_{u_j,u_{q+1}} > 0$ for each q.

Now, since D is locally nilpotent, it follows that

$$\begin{aligned} 0 &= D^l(x_1x_2x_3x_4) \\ &= \sum_{v_1=1}^{4} \cdots \sum_{v_l=1}^{4} \left(\prod_{q=0}^{l-1}(1 + \sum_{j=1}^{q} \delta_{u_j,u_{q+1}}) \right) \kappa_{v_1}\kappa_{v_2} \cdots \kappa_{v_l} x_1x_2x_3x_4 \mathbf{x}^{\delta_{v_1}+\delta_{v_2}+\cdots+\delta_{v_l}} \end{aligned} \tag{3.7}$$

for some $l > 0$. Set

$$F = \sum_{v_1=1}^{4} \cdots \sum_{v_l=1}^{4} \left(\prod_{q=0}^{l-1}(1 + \sum_{j=1}^{q} \delta_{u_j,u_{q+1}}) \right) \kappa_{v_1}\kappa_{v_2} \cdots \kappa_{v_l} y_{v_1}y_{v_2} \cdots y_{v_l}, \tag{3.8}$$

where $y_1, \ldots, y_4$ are indeterminates over k. Note that $\prod_{q=0}^{l-1}(1 + \sum_{j=1}^{q} \delta_{v_j,v_{q+1}})$ is a non-negative integer for any $1 \leq v_1, \ldots, v_l \leq 4$. So, by (3.6), the monomial $y_{u_1} \cdots y_{u_l}$ appears in F with its coefficient $\kappa_{u_1} \cdots \kappa_{u_l}$ multiplied by a nonzero integer. Since F is sent to zero by the substitution $y_i \mapsto \mathbf{x}^{\delta_i}$, there exists a monomial $y_1^{a_1} \cdots y_4^{a_4} \neq y_{u_1} \cdots y_{u_l}$ in F with $a_1 + \cdots + a_4 = l$ such that

$$a_1\delta_1 + \cdots + a_4\delta_4 = \delta_{u_1} + \cdots + \delta_{u_l}. \tag{3.9}$$

By adding $-l\delta_1$ to the both sides of (3.9), we get a nontrivial linear relation on $\delta_i - \delta_1$ for $i = 2, 3, 4$. This implies that $\{\delta_1, \ldots, \delta_4\}$ is contained in some two dimensional affine subspace of $\mathbf{R}^4$. This is a contradiction. Therefore, the proof is completed. □

Remark: It is possible that, for $k[x_1, x_2, x_3, x_4]$, any monomial derivation which is locally nilpotent is necessarily triangular in some coordinate system.

For $0 \leq r \leq n$, the k-derivation D on $k[\mathbf{x}]$ is called an *elementary* $k[x_1, \ldots, x_r]$-*derivation* if $D(x_i) \in k[x_1, \ldots, x_r]$ for all i, and $D(x_i) = 0$ for $1 \leq i \leq r$. We say that D is *elementary* if it is an elementary $k[x_1, \ldots, x_r]$-derivation for some r. By definition, an elementary derivation is triangular. The following fact is known for the kernel of an elementary derivation.

Theorem 3.6 (van den Essen and Janssen [6]) *Assume that D is an elementary $k[x_1, \ldots, x_r]$-derivation on $k[\mathbf{x}]$. If $r \leq 2$ or $n - r \leq 2$, then $k[\mathbf{x}]^D$ is finitely generated.*

For $n = 7$, there exists an elementary monomial $k[x_1, x_2, x_3]$-derivation whose kernel is not finitely generated (See [4, 8, 15]). However, we have the following for $n = 6$.

Corollary 3.7 *Assume that $n = 6$. If D is elementary and monomial, then $k[\mathbf{x}]^D$ is finitely generated.*

Proof. If $r \geq 3$, then $k[\mathbf{x}]^D$ is finitely generated by Corollary 3.3. If $r \leq 2$, then the assertion follows by Theorem 3.6. $\square$

We note that Khoury showed a stronger result that, under the assumption in Corollary 3.7, $k[\mathbf{x}]^D$ is generated by at most six elements [7].

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Chapter 2

A SAGBI basis for the kernel of a derivation

A SAGBI basis for the kernel of a derivation

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Abstract

We consider the kernel of a derivation on a polynomial ring, and give a criterion for it has a finite universal SAGBI basis. Furthermore, we show that the kernel of a triangular derivation with at most two dimensional support has a universal SAGBI basis with at most n elements by giving it explicitly.

1 Introduction

The SAGBI bases are the sets of generators of a subalgebra of a polynomial ring which have certain computational property. These are analogues of the Gröbner bases for an ideal of a polynomial ring. The term "SAGBI" is the abbreviation of "Subalgebra Analogue to Gröbner Bases for Ideals". This notion was introduced at the end of 1980's, independently, by Robbiano and Sweedler [9], and Kapur and Madlener [2]. This is applied for computing invariants. In [10], Stillman and Tsai invented an algorithm to compute invariants by using SAGBI basis. It enabled us to compute some of them.

Compared with the theory of Gröbner bases, however, that of SAGBI bases has made a slow progress. One of the factor is that a finitely generated subalgebra does not necessarily have a finite SAGBI basis. Therefore, it is a fundamental problem to consider

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when a subalgebra of a polynomial ring has a finite SAGBI basis. In this paper, we will work on this problem for the kernel of a derivation on a polynomial ring.

The kernel of a derivation on a polynomial ring is important in the study of invariant theory and the fourteenth problem of Hilbert. Recently, many researches on this object have been done, and we have a remarkable progress in this field. We believe that a computational methods will give us a further progress.

In this paper, k is always a field of characteristic zero. Let $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, and $k[\mathbf{x}, \mathbf{x}^{-1}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ the Laurent polynomial ring in n variables over k. For each $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n$, we denote by $\mathbf{x}^a$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$. Let us denote by Ω the set of total orders $\preceq$ on $\mathbf{Z}^n$ such that $a \preceq b$ implies $a + c \preceq b + c$ for any $a, b, c \in \mathbf{Z}^n$, and by Ω_0 the set of $\preceq \in \Omega$ such that zero vector is the minimum among ${\mathbf{Z}_{\geq 0}}^n$ for the order relation $\preceq$. An element of Ω_0 is called a *monomial order*. When an order $\preceq$ is given, we write $a \prec b$ if $a \preceq b$ and $a \neq b$ for $a, b \in \mathbf{Z}^n$.

Let $\preceq$ be an element of Ω. For $f = \sum_{a \in \mathbf{Z}^n} \mu_a \mathbf{x}^a \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we define the *support* $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) = \{a \mid \mu_a \neq 0\}. \tag{1.1}$$

If $f \neq 0$, then set $v_{\preceq}(f)$ to the maximum element of $\mathrm{supp}(f)$ for $\preceq$. The maximum exists, since $\mathrm{supp}(f)$ is a nonempty finite subset of $\mathbf{Z}^n$. For any $f, g \in k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$, it follows that $v_{\preceq}(fg) = v_{\preceq}(f) + v_{\preceq}(g)$. We define the *initial term* $\mathrm{in}_{\preceq}(f)$ of f by

$$\mathrm{in}_{\preceq}(f) = \mu_{v_{\preceq}(f)} \mathbf{x}^{v_{\preceq}(f)} \tag{1.2}$$

if $f \neq 0$, while we define $\mathrm{in}_{\preceq}(0) = 0$. Then, it follows that

$$\mathrm{in}_{\preceq}(fg) = \mathrm{in}_{\preceq}(f)\,\mathrm{in}_{\preceq}(g) \tag{1.3}$$

for any $f, g \in k[\mathbf{x}, \mathbf{x}^{-1}]$. For a k-vector subspace $V \subset k[\mathbf{x}]$, we define the *initial vector space* $\mathrm{in}_{\preceq}(V)$ to be the k-vector space generated by $\{\mathrm{in}_{\preceq}(f) \mid f \in V\}$. If $A \subset k[\mathbf{x}]$ is a k-subalgebra of $k[\mathbf{x}]$, then $\mathrm{in}_{\preceq}(A)$ is a k-algebra. It is called the *initial algebra* of A. A subset $\mathcal{S} \subset A$ is said to be a *SAGBI basis* for A if A is generated by $\mathcal{S}$ over k, and

$$\mathrm{in}_{\preceq}(A) = k[\{\mathrm{in}_{\preceq}(s) \mid s \in \mathcal{S}\}]. \tag{1.4}$$

We say that $\mathcal{S} \subset A$ is a *universal SAGBI basis* for A if it is a SAGBI basis for A with respect to any $\preceq \in \Omega$.

We remark that, for any $\preceq \in \Omega_0$, the condition (1.4) implies that A is generated by $\mathcal{S}$, i.e., $\mathcal{S}$ is a SAGBI basis for A (See [9, Proposition 1.16]). The condition (1.4) is equivalent to the condition that the subsemigroup $\{v_{\preceq}(f) \mid f \in A \setminus \{0\}\}$ of $\mathbf{Z}^n$ is generated by $\{v_{\preceq}(f) \mid s \in \mathcal{S} \setminus \{0\}\}$. Hence, $\mathcal{S}$ is a universal SAGBI basis for A if and only if the subsemigroup $\{v_{\preceq}(f) \mid f \in A\setminus\{0\}\}$ of $\mathbf{Z}^n$ is generated by $\{v_{\preceq}(f) \mid s \in \mathcal{S}\setminus\{0\}\}$ for any $\preceq \in \Omega$.

By definition, there exist the following implications:

$$\begin{gathered}
A \text{ has a finite universal SAGBI basis.} \\
\Downarrow \\
A \text{ has a finite SAGBI basis for some } \preceq \in \Omega. \\
\Downarrow \\
A \text{ is finitely generated over } k.
\end{gathered}$$

However, the converse of each implication is not true, in general. Actually, Robbiano and Sweedler showed that, for $A = k[x_1, x_1x_2 - x_2^2, x_1x_2^2]$, the set $\{x_1, x_1x_2 - x_2^2, x_1x_2^2\}$ is a SAGBI basis for $\preceq \in \Omega$ with $x_1 \prec x_2$, but A does not have a finite SAGBI basis for $\preceq \in \Omega$ with $x_2 \prec x_1$ ([9, Example 4.11]). There is also an example of an finitely generated subalgebra which does not have finite SAGBI basis for any $\preceq \in \Omega$ (See for example Chapter 4). In the present paper, we will give a sufficient condition on the derivation that the kernel has a finite universal SAGBI basis.

For a commutative ring R and an R-algebra A, a R-homomorphism $D : A \to A$ is called an R-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. For an R-submodule $V \subset A$, we denote by

$$V^D = \{f \in V \mid D(f) = 0\}. \tag{1.5}$$

If V is a R-subalgebra of A, then V^D is a R-subalgebra of V. We will consider the kernel $k[\mathbf{x}]^D$ of a k-derivation D on $k[\mathbf{x}]$. The k-subalgebra $k[\mathbf{x}]^D$ of $k[\mathbf{x}]$ is not necessarily finitely generated (cf. Chapter 3). We note that this is a kind of the fourteenth problem of Hilbert.

We define the *support* $\operatorname{supp}(D)$ of D by

$$\operatorname{supp}(D) = \bigcup_{i=1}^{n} \operatorname{supp}(x_i^{-1} D(x_i)). \tag{1.6}$$

Then, there exists $\kappa_{\delta,i} \in k$ for each δ such that $x_i^{-1}D(x_i) = \sum_{\delta \in \operatorname{supp}(D)} \kappa_{\delta,i}\mathbf{x}^\delta$ for $1 \le i \le n$. Define a homomorphism $\lambda_\delta : \mathbf{Z}^n \to k$ of additive groups by

$$\lambda_\delta((a_1, \dots, a_n)) = a_1\kappa_{\delta,1} + \dots + a_n\kappa_{\delta,n} \tag{1.7}$$

for each δ. We define a subset $\operatorname{supp}^\circ(D) \subset \operatorname{supp}(D)$ as follows. Set $S_0 = \operatorname{supp}(D)$ and

$$S_{i+1} = \{\delta \in S_i \mid \delta' - \delta \notin \ker \lambda_\delta \text{ for some } \delta' \in S_i\} \tag{1.8}$$

for each $i \in \mathbf{Z}_{\ge 0}$, inductively. Then, define $\operatorname{supp}^\circ(D)$ by the set of $\delta \in \operatorname{supp}(D)$ contained in the convex hull of $\bigcap_{i=0}^\infty S_i$ in $\mathbf{R}^n$.

In [Chapter 1, Theorem 1.2], we showed that $k[\mathbf{x}]^D$ is finitely generated over k if the dimension of $\operatorname{supp}(D)$ is at most two. The following result generalizes this.

Theorem 1.1 *Assume that D is a k-derivation on $k[\mathbf{x}]$. If the dimension of* $\operatorname{supp}^\circ(D)$ *is at most two, then $k[\mathbf{x}]^D$ has a finite universal SAGBI basis.*

Here, for a subset $S \subset \mathbf{R}^n$, we define the dimension $\dim S$ of S by the dimension of the $\mathbf{R}$-vector subspace of $\mathbf{R}^n$ generated by $\{s - t \mid s, t \in S\}$ if $S \ne \emptyset$, and by -1 if $S = \emptyset$. Since $\operatorname{supp}^\circ(D)$ can not be a single point, $\dim \operatorname{supp}^\circ(D) \ne 0$ for any D.

A k-derivation D on $k[\mathbf{x}]$ is said to be *triangular* if $D(x_i)$ is in $k[x_1, \dots, x_{i-1}]$ for each i. In this case, we know the following.

Theorem 1.2 *Assume that D is a triangular k-derivation on $k[\mathbf{x}]$. If the dimension of* $\operatorname{supp}^\circ(D)$ *is at most two, then there exists a universal SAGBI basis for $k[\mathbf{x}]^D$ with at most n elements.*

Moreover, we describe the universal SAGBI basis explicitly.

In Section 2, we review the proof of [Chapter 1, Lemma 2.1] first. Then, we prove Theorem 2.2 which guarantees the existence of a finite universal SAGBI basis for the kernel of some derivation. We deduce Theorem 1.1 from this theorem. Theorem 1.2 will be proved in Section 3.

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2 Finite universal SAGBI bases

Let A be a finitely generated normal domain over k, and K the field of fractions of A. We assume that K is a regular extension of k, i.e., $K \otimes_k \bar{k}$ is a field for the algebraic closure $\bar{k}$ of k. In [Chapter 1, Lemma 2.1], we showed the following. Let L be a subfield of K containing k, and $g_1, \ldots, g_r$ be elements of $K \setminus \{0\}$. Then, the k-subalgebra

$$R = \sum_{i_1, \ldots, i_r \in \mathbf{Z}} \left(L g_1^{i_1} \cdots g_r^{i_r} \cap A \right) \tag{2.1}$$

of A is finitely generated over k if L is a simple extension of k. We have a stronger statement as follows.

Lemma 2.1 *Assume that $L = k(u_0/u_1)$ for some $u_0, u_1 \in A$. Then, we may find a finite subset $\Sigma_0 \subset \mathbf{P}^1_{\bar{k}}$ of closed points such that, for any finite subset $\Sigma \subset \mathbf{P}^1_{\bar{k}}$ of closed points containing Σ_0, there exist $f_1, \ldots, f_s \in R \otimes_k \bar{k}$ with the following property. Assume that f is in $L g_1^{i_1} \cdots g_r^{i_r} \cap A$ for some $i_1, \ldots, i_r \in \mathbf{Z}$. Then, there exists $h \in \bar{k}[u_0, u_1] \setminus \{0\}$ of the form*

$$h = \prod_{j=1}^{q} (\alpha_j u_0 - \beta_j u_1)^{m_j} \tag{2.2}$$

with $(\alpha_j : \beta_j) \in \mathbf{P}^1_{\bar{k}} \setminus \Sigma$ and $m_j \in \mathbf{Z}_{\geq 0}$ for $j = 1, \ldots, q$ such that $u_0^i u_1^{m-i} f/h$ is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$ for $0 \leq i \leq m$, where $m = \sum_{j=1}^{q} m_j$.

Proof. We set $\bar{L} = L \otimes_k \bar{k}$, $\bar{A} = A \otimes_k \bar{k}$, $\bar{K} = A \otimes_k \bar{K}$ and $\bar{R} = R \otimes_k \bar{k}$. First, assume that u_0/u_1 is transcendental over k. Let $\phi : \operatorname{Spec} \bar{A} \dashrightarrow \mathbf{P}^1_{\bar{k}}$ be the dominant rational map defined by the inclusion map $\iota : \bar{L} \to \bar{K}$. Then, we may consider the homomorphism

$$\phi^* : \operatorname{Div}(\mathbf{P}^1_{\bar{k}}) \to \operatorname{Div}(\operatorname{Spec} \bar{A}) \tag{2.3}$$

of the divisor groups of $\mathbf{P}^1_{\bar{k}}$ and $\operatorname{Spec} \bar{A}$. Since the complement of the image of ϕ is a finite set, $\ker \phi^*$ is finitely generated. In the proof of [Chapter 1, Lemma 2.1], we showed the following.

There exists a finite subset $\Sigma \subset \mathbf{P}^1_{\bar{k}}$ of closed points of as follows:
(i) $\ker \phi^*$ is contained in the subgroup of $\operatorname{Div}(\mathbf{P}^1_{\bar{k}})$ generated by Σ, where we regard Σ as a set of prime divisors.
(ii) Let p be the generic point of a prime divisor which appears in $(g_i) \in \operatorname{Div}(\operatorname{Spec} \bar{A})$ for some $1 \leq i \leq r$. Then, $\phi(p)$ is in Σ, unless it is the generic point of $\mathbf{P}^1_{\bar{k}}$.

If Σ is a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points as above, then there exist finite elements $f_1, \ldots, f_s \in \bar{R}$ with the following property. Assume that f is an element of $\bar{L} g_1^{a_1} \cdots g_r^{a_r} \cap \bar{A} \setminus \{0\}$ for some $(a_i)_i \in \mathbf{Z}^r$ such that the supports of zeros and poles of the rational function $f/g_1^{a_1} \cdots g_r^{a_r}$ on $\mathbf{P}^1_{\bar{k}}$ are contained in Σ. Then, f is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$.

Let Σ_0 be a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points satisfying (i) and (ii) which contains the supports of zeros and poles of u_0/u_1. We show that it satisfies the desired property. Assume that Σ is a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points containing Σ_0. Then, Σ also satisfies (i) and (ii). Hence, there exist finite elements $f_1, \ldots, f_s \in \bar{R}$ as above. Assume that f is in $L g_1^{i_1} \cdots g_r^{i_r} \cap A \setminus \{0\}$. Put $h' = f/(g_1^{i_1} \cdots g_r^{i_r})$, and set $(h') = \sum_{p \in \mathbf{P}^1_{\bar{k}}} m_p p$ and $E = \sum_{p \in \Sigma} m_p p$. Note that h' is in $H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E))$. Hence,

$$h = \prod_{p \in \mathbf{P}^1_{\bar{k}} \setminus \Sigma} (\alpha_p u_0 - \beta_p u_1)^{m_p} \tag{2.4}$$

is an element of $\bar{k}[u_0, u_1] \setminus \{0\}$. Here, for each closed point $p \in \mathbf{P}^1_{\bar{k}}$, we assign $(\alpha_p, \beta_p) \in \bar{k}^2 \setminus \{0\}$ so that $h \prod_{p \in \mathbf{P}^1_{\bar{k}}} (\alpha_p u_0 - \beta_p u_1)^{-m_p} \in \bar{k} \setminus \{0\}$ for each $h \in \bar{L}^\times$ with $(h) = \sum_{p \in \mathbf{P}^1_{\bar{k}}} m_p p$, and identify p with the ratio $(\alpha_p : \beta_p)$. We set $m = \sum_{p \in \mathbf{P}^1_{\bar{k}} \setminus \Sigma} m_p$. Then, the supports of zeros and poles of $u_0^i u_1^{m-i} h'/h$ are contained in Σ for each $0 \leq i \leq m$. Note that

$$H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E)) g_1^{i_1} \cdots g_r^{i_r} \subset \bar{L} g_1^{i_1} \cdots g_r^{i_r} \cap \bar{A}. \tag{2.5}$$

Since $u_0^i u_1^{m-i} h'/h$ is in $H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E))$, we have $u_0^i u_1^{m-i} f/h \in \bar{L} g_1^{i_1} \cdots g_r^{i_r} \cap \bar{A}$. Hence, $u_0^i u_1^{m-i} f/h$ is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$ by assumption. Thus, the assertion is true if u_0/u_1 is transcendental over k.

Now, assume that u_0/u_1 is algebraic over k. Then, $L = k$, since K/k is a regular extension. In this case, the proof of [Chapter 1, Lemma 2.1] shows that there exist finite elements $f_1, \ldots, f_s \in \bar{R}$ such that every element of $\bar{k} g_1^{i_1} \cdots g_r^{i_r} \cap \bar{A}$ is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$. Hence, the assertion holds for $\Sigma_0 = \emptyset$ and $h = 1$. Therefore, the proof of Lemma 2.1 is completed. □

Now, let Γ be an additive group, $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ a Γ-graded finitely generated normal k-subalgebra of $k[\mathbf{x}]$, and D a k-derivation defined on an extension of A. Here, we say that a k-algebra C is Γ-graded if $C = \bigoplus_{\gamma \in \Gamma} C_\gamma$ for some k-vector spaces $C_\gamma \subset C$ such that $C_\gamma C_\mu \subset C_{\gamma + \mu}$ for every $\gamma, \mu \in \Gamma$. An element f of C is said to be Γ-homogeneous if f is in C_γ for some $\gamma \in \Gamma$, and this γ is called the Γ-degree of f. Assume that $A^D = \bigoplus_{\gamma \in \Gamma} A^D_\gamma$ and $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ for any $f \in A^D_\gamma, g \in A^D_\mu$ with $\gamma \neq \mu$. Since A is a domain, the set $A_H = \bigcup_{\gamma \in \Gamma} A_\gamma \setminus \{0\}$ of nonzero Γ-homogeneous elements is multiplicatively closed. We set $B = A_H^{-1} A$ the localization of A by A_H. Then, the Γ-grading $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$ is defined by setting

$$B_\gamma = \{f/g \mid (f, g) \in A_{\gamma' + \gamma} \times (A_{\gamma'} \setminus \{0\}) \text{ for some } \gamma' \in \Gamma\} \tag{2.6}$$

for each $\gamma \in \Gamma$. Note that B_0 is a field containing k.

Theorem 2.2 *If* $\mathrm{trans.deg}_k B_0^D \leq 1$, *then* A^D *has a finite universal SAGBI basis.*

Proof. Take $\preceq \in \Omega$ arbitrarily. Then, we have $\mathrm{in}_{\preceq}(A^D) = \bigoplus_{\gamma\in\Gamma} \mathrm{in}_{\preceq}(A^D_\gamma)$, since the supports of Γ-homogeneous elements in A^D with distinct Γ-degrees do not intersect by assumption. By the remark after the definition of the universal SAGBI bases, it suffices to show the existence of such elements $f_1, \ldots, f_t \in A^D$ that, for any Γ-homogeneous element $f \in A^D$, there exist $a_1, \ldots, a_t \in \mathbf{Z}_{\geq 0}$ such that $v_{\preceq}(f) = a_1 v_{\preceq}(f_1) + \cdots + a_t v_{\preceq}(f_1)$.

Since $\mathrm{trans.deg}_k\, B^D_0 \leq 1$, the field B^D_0 is a simple extension of k. Actually, Lüroth's theorem says that B^D_0 is a rational function field of one variable over k if $\mathrm{trans.deg}_k\, B^D_0 = 1$, while $B^D_0 = k$ otherwise. Let $u_0, u_1 \in A$ be Γ-homogeneous elements with $B^D_0 = k(u_0/u_1)$. Then, we may find a finite subset $\Sigma_1 \subset \mathbf{P}^1_{\bar{k}}$ of closed points such that, for any finite subset $\Sigma \subset \mathbf{P}^1_{\bar{k}}$ of closed points containing Σ_1, the convex hull of $\mathrm{supp}(\alpha_1 u_0 - \alpha_0 u_1)$ in $\mathbf{R}^n$ are the same for any $(\alpha_0, \alpha_1) \in \bar{k}^2 \setminus \{0\}$ with $(\alpha_0 : \alpha_1) \notin \Sigma$. If this is the case, then it follows that

$$v_{\preceq}(\alpha_1 u_0 - \alpha_0 u_1) = v_{\preceq}(u_e) \tag{2.7}$$

for all $(\alpha_0, \alpha_1) \in \bar{k}^2 \setminus \{0\}$ with $(\alpha_0 : \alpha_1) \notin \Sigma$ for some $e \in \{0, 1\}$.

By a similar argument as in the proof of [Chapter 1, Theorem 2.1], we may find Γ-homogeneous elements $g_1, \ldots, g_r \in B^D$ such that, for each $\gamma \in \Gamma$, there exist $i_1, \ldots, i_r \in \mathbf{Z}$ such that $A^D_\gamma = B^D_0 g_1^{i_1} \cdots g_r^{i_r} \cap A$. Since $A^D = \bigoplus_{\gamma\in\Gamma} A^D_\gamma$, we get

$$A^D = \sum_{i_1,\ldots,i_r \in \mathbf{Z}} \left(B^D_0 g_1^{i_1} \cdots g_r^{i_r} \cap A \right). \tag{2.8}$$

By Lemma 2.1, there exist a finite subset $\Sigma \subset \mathbf{P}^1_{\bar{k}}$ of closed points containing Σ_1 and finite elements $f'_1, \ldots, f'_s$ in $k[\mathbf{x}]^D \otimes_k \bar{k}$ which satisfy the following property. Assume that $f \in A^D \setminus \{0\}$ is a Γ-homogeneous element. Then, there exists $h \in \bar{k}[u_0, u_1] \setminus \{0\}$ of the form $h = \prod_{j=1}^q (\alpha_j u_0 - \beta_j u_1)^{m_j}$ with $(\alpha_j : \beta_j) \in \mathbf{P}^1_{\bar{k}} \setminus \Sigma$ and $m_j \in \mathbf{Z}_{\geq 0}$ for $j = 1, \ldots, q$ such that $u_0^i u_1^{m-i} f/h$ is equal to a product of powers of $f'_1, \ldots, f'_s$ multiplied by an element of $\bar{k} \setminus \{0\}$ for $0 \leq i \leq m$, where $m = \sum_{j=1}^q m_j$.

Choose $f_1, \ldots, f_t \in A^D$ so that $\bar{k} f_1 + \cdots + \bar{k} f_t$ contains $f'_1, \ldots, f'_s$ and there exists $1 \leq l \leq t$ such that $v_{\preceq}(f'_j) = v_{\preceq}(f_l)$ for each $1 \leq j \leq s$. We show that these are the elements which we are looking for. It follows that

$$\begin{aligned} v_{\preceq}(f) &= v_{\preceq}\bigl((f/h)\prod_{j=1}^{q}(\alpha_j u_0 - \beta_j u_1)^{m_j}\bigr) && (2.9) \\ &= v_{\preceq}(f/h) + \sum_{j=1}^{q} m_j v_{\preceq}(\alpha_j u_0 - \beta_j u_1) && (2.10) \\ &= v_{\preceq}(f/h) + \sum_{j=1}^{q} m_j v_{\preceq}(u_e) && (2.11) \\ &= v_{\preceq}(u_e^m f/h). && (2.12) \end{aligned}$$

Then, there exist $a'_1, \dots, a'_s \in \mathbf{Z}_{\geq 0}$ such that $u_e^m f/h$ is equal to $(f'_1)^{a'_1} \cdots (f'_s)^{a'_s}$ up to a scalar multiplication. By the choice of $f_1, \dots, f_t$, we have $\sum_{i=1}^{s} a'_i v_{\preceq}(f'_i) = \sum_{i=1}^{t} a_i v_{\preceq}(f_i)$ for some $a_1, \dots, a_t \in \mathbf{Z}_{\geq 0}$. Therefore, $v_{\preceq}(f) = \sum_{i=1}^{t} a_i v_{\preceq}(f_i)$. This completes the proof. $\square$

Now, we show Theorem 1.1 as a consequence of Theorem 3.6. To show this theorem, we need the following lemma. Let D be a k-derivation on $k[\mathbf{x}]$. For each $\delta \in \operatorname{supp}(D)$, we define

$$D_\delta = \mathbf{x}^\delta \left(\kappa_{\delta,1} x_1 \frac{\partial}{\partial x_1} + \cdots + \kappa_{\delta,n} x_n \frac{\partial}{\partial x_n} \right). \qquad (2.13)$$

Then, it follows that

$$D_\delta(\mathbf{x}^a) = \lambda_\delta(a) \mathbf{x}^{a+\delta} \qquad (2.14)$$

for any $a \in \mathbf{Z}^n$. For a subset $S \subset \operatorname{supp}(D)$, we set $D_S = \sum_{\delta \in S} D_\delta$.

Lemma 2.3 *Assume that D is a k-derivation on $k[\mathbf{x}]$. We set*

$$\mathcal{S} = \bigl\{ a \in {\mathbf{Z}_{\geq 0}}^n \mid a \in \ker \lambda_\delta \textit{ for all } \delta \in \operatorname{supp}(D) \setminus \operatorname{supp}^\circ(D) \bigr\}. \qquad (2.15)$$

Then, it follows that $k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}$, where $D^\circ = D_{\operatorname{supp}^\circ(D)}$.

Proof. We will prove by induction on the number of elements of $\operatorname{supp}(D)$. Put $S = \operatorname{supp}(D)$. If $S \neq \operatorname{supp}^\circ(D)$, then there exists $\delta \in S \setminus \operatorname{supp}^\circ(D)$ which is a vertex of the convex hull of S in $\mathbf{R}^n$ such that $S + \{-\delta\} \subset \ker \lambda_\delta$. Then, it suffices to show that

$$k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in {\mathbf{Z}_{\geq 0}}^n \cap \ker \lambda_\delta\}]^{D_{S \setminus \{\delta\}}} \qquad (2.16)$$

by the following reason. Note that

$$k[\{\mathbf{x}^a \mid a \in \mathbf{Z}_{\geq 0}{}^n \cap \ker \lambda_\delta\}]^{D_{S\setminus\{\delta\}}} = k[\{\mathbf{x}^a \mid a \in \mathbf{Z}_{\geq 0}{}^n \cap \ker \lambda_\delta\}] \cap \left(k[\mathbf{x}]^{D_{S\setminus\{\delta\}}}\right). \tag{2.17}$$

Since $\operatorname{supp}^\circ(D_{S\setminus\{\delta\}}) = \operatorname{supp}^\circ(D)$, we get $k[\mathbf{x}]^{D_{S\setminus\{\delta\}}} = k[\{\mathbf{x}^a \mid a \in \mathcal{S}'\}]^{D^\circ}$ by induction assumption, where

$$\mathcal{S}' = \left\{a \in \mathbf{Z}_{\geq 0}{}^n \mid a \in \ker \lambda_{\delta'} \text{ for all } \delta' \in S \setminus \left(\{\delta\} \cup \operatorname{supp}^\circ(D)\right)\right\}. \tag{2.18}$$

On the other hand, we have

$$k[\{\mathbf{x}^a \mid a \in \mathbf{Z}_{\geq 0}{}^n \cap \ker \lambda_\delta\}] \cap \left(k[\{\mathbf{x}^a \mid a \in \mathcal{S}'\}]^{D^\circ}\right) = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}. \tag{2.19}$$

Therefore, $k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}$ follows from (2.16).

Assume that f is in the right hand side of (2.16). Then, $D_\delta(f) = 0$ by (2.14). Hence, we have $D(f) = D_\delta(f) + D_{S\setminus\{\delta\}}(f) = 0$. Namely, f is in $k[\mathbf{x}]^D$. Thus, the right hand side of (2.16) is contained in the left.

For the converse, we take any $f \in k[\mathbf{x}]^D$, and show that it is in the right hand side of (2.16). Recall that $k[\mathbf{x}]^D$ has a $\mathbf{Z}^n/M$-grading structure, where $M = \sum_{\delta' \in S} \mathbf{Z}(\delta' - \delta)$ (cf. Section 3 of Chapter 1). So, we may assume that f is $\mathbf{Z}^n/M$-homogeneous. Choose $\preceq \in \Omega$ so that δ is the maximum among S for $\preceq$. Then, $v_\preceq(f)$ is in $\ker \lambda_\delta$. We show this by contradiction. If $\lambda_\delta(v_\preceq(f)) \neq 0$, then $D_\delta(\mathrm{in}_\preceq(f)) \neq 0$ by (2.14). Since $D(f) = 0$, the term $D_\delta(\mathrm{in}_\preceq(f))$ is eliminated in the expression

$$D(f) = D_\delta(\mathrm{in}_\preceq(f)) + D_\delta(f - \mathrm{in}_\preceq(f)) + D_{S\setminus\{\delta\}}(f). \tag{2.20}$$

Since $\operatorname{supp}(D(f))$ is contained in $\operatorname{supp}(D) + \operatorname{supp}(f)$, there exist $\delta' \in \operatorname{supp}(D)$ and $a' \in \operatorname{supp}(f)$ such that $\delta' + a' = \delta + v_\preceq(f)$ and $\delta' \neq \delta$ or $a' \neq v_\preceq(f)$. Since $\delta' \prec \delta$ and $a' \prec v_\preceq(f)$, this is a contradiction. Thus, $v_\preceq(f)$ is in $\ker \lambda_\delta$.

Since f is $\mathbf{Z}^n/M$-homogeneous, $\operatorname{supp}(f) + \{-v_\preceq(f)\}$ is contained in M. On the other hand, M is contained in $\ker \lambda_\delta$, since $S + \{-\delta\} \subset \ker \lambda_\delta$. It implies $\operatorname{supp}(f) \subset \ker \lambda_\delta$, since $v_\preceq(f) \in \ker \lambda_\delta$. So, we have $D_\delta(f) = 0$ by (2.14). Hence,

$$D_{S\setminus\{\delta\}}(f) = D_{S\setminus\{\delta\}}(f) + D_\delta(f) = D(f) = 0. \tag{2.21}$$

This shows that f is contained in the right hand side of (2.16). Thus, we get the equality (2.16). This completes the proof. $\square$

The proof of Theorem 1.1. We set $A = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]$, $D^\circ = D_{\mathrm{supp}^\circ(D)}$ and $S^\circ = \mathrm{supp}^\circ(D)$. Then, A is a finitely generated normal k-subalgebra of $k[\mathbf{x}]$, since $\mathcal{S}$ is a finitely generated normal subsemigroup of $\mathbf{Z}_{\geq 0}{}^n$. Here, we say that a subsemigroup $\mathcal{S}$ of $\mathbf{Z}^n$ is normal if $\mathcal{S} = \sum_{s\in\mathcal{S}} \mathbf{Z}s \cap \sum_{s\in\mathcal{S}} \mathbf{R}_{\geq 0}s$.

We put $\Gamma = \mathbf{Z}^n/M^\circ$, where $M^\circ = \sum_{\delta'\in S^\circ} \mathbf{Z}(\delta' - \delta)$. Consider the Γ-grading structure $k[\mathbf{x}] = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma$ on $k[\mathbf{x}]$. Then, it follows that

$$k[\mathbf{x}]^{D^\circ} = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma^{D^\circ}. \tag{2.22}$$

We set Γ' the image of

$$\mathcal{M} = \left\{a \in \mathbf{Z}^n \mid a \in \ker \lambda_\delta \text{ for all } \delta \in \mathrm{supp}(D) \setminus \mathrm{supp}^\circ(D)\right\} \tag{2.23}$$

in Γ. Then, it follows that $A = \bigoplus_{\gamma\in\Gamma'} k[\mathbf{x}]_\gamma$. Actually, for $a \in \mathbf{Z}_{\geq 0}{}^n$, we have $\mathbf{x}^a \in \bigoplus_{\gamma\in\Gamma'} k[\mathbf{x}]_\gamma$ if and only if $a \in \mathcal{M} + M^\circ$. Since $M^\circ \subset \mathcal{M}$ and $\mathcal{S} = \mathcal{M} \cap \mathbf{Z}_{\geq 0}{}^n$, it is equivalent to $a \in \mathcal{S}$. Thus, $A = \bigoplus_{\gamma\in\Gamma'} k[\mathbf{x}]_\gamma$. In particular, $A^{D^\circ} = \bigoplus_{\gamma\in\Gamma'} k[\mathbf{x}]_\gamma^{D^\circ}$ by (2.22).

Let $B = \bigoplus_{\gamma\in\Gamma'} B_\gamma$ denote the localization of A by $\bigcup_{\gamma\in\Gamma'} k[\mathbf{x}]_\gamma \setminus \{0\}$, and $k(M^\circ)$ the subfield of the quotient field of $k[\mathbf{x}]$ generated by $\{\mathbf{x}^a \mid a \in M^\circ\}$ over k. Then, $B_0 \subset k(M^\circ)$ and $\mathrm{trans.deg}_k\, k(M^\circ) \leq 2$. Assume that $\mathrm{trans.deg}_k\, k(M^\circ)^{D^\circ} = 2$. Then, we get $k(M^\circ) = k(M^\circ)^{D^\circ}$ by a similar argument as in the proof of [Chapter 1, Theorem 1.3]. In this case, the proof of [Chapter 1, Lemma 3.2] says that $k[\mathbf{x}]^{D^\circ} = k[\{\mathbf{x}^a \mid a \in \mathcal{S}'\}]$ for some finitely generated subsemigroup $\mathcal{S}'$ of $\mathbf{Z}_{\geq 0}{}^n$. Hence, $A^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S} \cap \mathcal{S}'\}]$. Since the semigroup $\mathcal{S} \cap \mathcal{S}'$ is finitely generated by Gordan's lemma [8, Proposition 1.1.(ii)], A^D is generated by a finite set of monomials over k. This set is a universal SAGBI basis for A^D. If $\mathrm{trans.deg}_k\, k(M^\circ)^{D^\circ} \leq 1$, then $\mathrm{trans.deg}_k\, B_0^{D^\circ} \leq 1$. Hence, the assertion of Theorem 1.1 follows from Theorem 2.2. Thus, the proof is completed. $\square$

The dimension of $\mathrm{supp}^\circ(D)$ is one of the measure which shows the "complexity" of $k[\mathbf{x}]^D$. If $\mathrm{supp}^\circ(D) = \emptyset$, then $k[\mathbf{x}]^D$ is a semigroup ring of a finitely generated normal subsemigroup of ${\mathbf{Z}_{\geq 0}}^n$.

We say that a k-derivation D on a k-algebra B is *locally nilpotent* if, for each $f \in B$, there exists $r \in \mathbf{Z}_{\geq 0}$ such that $D^r(f) = 0$. We see easily that a triangular derivation on $k[\mathbf{x}]$ is locally nilpotent.

Proposition 2.4 *Assume that D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. Then, $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$ if and only if $\mathrm{supp}^\circ(D) = \emptyset$. If this is the case, we have $D = f\partial/\partial x_i$ for some $1 \leq i \leq n$ and $f \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \setminus \{0\}$.*

Proof. Since $D \neq 0$, it is clear that $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$ if $\mathrm{supp}^\circ(D) = \emptyset$.

Assume that $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$. Then, there exists $\delta \in \mathrm{supp}(D)$ which is a vertex of the convex hull of $\mathrm{supp}(D)$ such that $\mathrm{supp}(D) + \{-\delta\} \subset \ker \lambda_\delta$. First, we show that one component of δ is -1. By definition, components of each element of $\mathrm{supp}(D)$ are not less than -1. We suppose that $\delta \in {\mathbf{Z}_{\geq 0}}^n$, and show a contradiction. Let a be an element of ${\mathbf{Z}_{\geq 0}}^n \setminus \ker \lambda_\delta$ if $\lambda_\delta(\delta) = 0$, and $a = \delta$ otherwise. Then, it follows that $\lambda_\delta(a + j\delta) \neq 0$ for any $j \in \mathbf{Z}_{\geq 0}$. By a repeated use of (2.14), we get

$$D^l(\mathbf{x}^a) = \sum_{\delta_1 \in \mathrm{supp}(D)} \cdots \sum_{\delta_l \in \mathrm{supp}(D)} \left(\prod_{t=1}^{l} \lambda_{\delta_t}\Big(a + \sum_{j=1}^{t-1} \delta_j\Big) \right) \mathbf{x}^{a+\delta_1+\cdots+\delta_l}. \tag{2.24}$$

We set

$$\sum_{\delta_1 \in \mathrm{supp}(D)} \cdots \sum_{\delta_l \in \mathrm{supp}(D)} \left(\prod_{t=1}^{l} \lambda_{\delta_t}\Big(a + \sum_{j=1}^{t-1} \delta_j\Big) \right) y_{\delta_1} \cdots y_{\delta_l}, \tag{2.25}$$

where $\{y_{\delta'} \mid \delta' \in \mathrm{supp}(D)\}$ is the set of indeterminates over k. Since D is locally nilpotent, (2.24) is zero for sufficiently large l. For this l, (2.25) is sent to zero by the substitution $y_{\delta'} \mapsto \mathbf{x}^{\delta'}$ for $\delta' \in \mathrm{supp}(D)$. Since δ is a vertex of the convex hull of $\mathrm{supp}(D)$ in $\mathbf{R}^n$, we have $\delta_1 + \cdots + \delta_l = l\delta$ if and only if $\delta_1 = \cdots = \delta_l = \delta$. Hence, the coefficient of $\mathbf{x}^{a+l\delta}$ in $D^l(\mathbf{x}^a)$ is equal to $\prod_{j=0}^{l-1} \lambda_\delta(a + j\delta)$. However, it is not zero by the choice of a. This is a contradiction. Therefore, some component of δ is -1.

Without loss of generality, we may assume that the first component of δ is -1. Then, it follows that $\kappa_{\delta,1} \neq 0$ and $\kappa_{\delta,i} = 0$ for $2 \leq i \leq n$. Since $\mathrm{supp}(D) + \{-\delta\} \subset \ker \lambda_\delta$, the first component of every element of $\mathrm{supp}(D)$ is -1. Thus, we have $D = f\partial/\partial x_1$ for some $f \in k[x_2, \ldots, x_n]$. Moreover, $\lambda_{\delta'}(\delta'' - \delta''') = 0$ for any $\delta', \delta'', \delta''' \in \mathrm{supp}(D)$. This implies that $\mathrm{supp}^\circ(D) = \emptyset$. Thus, the proof is completed. □

3 A triangular derivation with two dimensional support

Maubach [7] and Khoury [3] studied in respective papers the kernels of some triangular derivations on $k[\mathbf{x}]$. They showed the finite generation of them by giving the generators explicitly. In this section, we consider the kernel $k[\mathbf{x}]^D$ of a triangular derivation D on $k[\mathbf{x}]$ with the dimension of $\mathrm{supp}^\circ(D)$ is at most two. We will determine a universal SAGBI basis for it explicitly. This implies the results of both Maubach and Khoury as special cases.

Assume that D is a nonzero triangular derivation on $k[\mathbf{x}]$. We set N_D the number of indices $i \in \{1, \ldots, n\}$ such that $D(x_i) \neq 0$. If $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$, then $N_D = n - 1$. Actually, $D = f\partial/\partial x_i$ for some i and $f \in k[\mathbf{x}]$ by Proposition 2.4. In this case, $\{x_j \mid j \neq i\}$ is a universal SAGBI basis for $k[\mathbf{x}]^D$. In case of $N_D = n - 2$, we will determine a universal SAGBI basis for $k[\mathbf{x}]^D$ with $n - 1$ elements explicitly in Corollary 3.5 below as a consequence of a fact on the kernel of a locally nilpotent derivation. Our main result of this section is for the case where $N_D \leq n - 3$.

Lemma 3.1 *Assume that $n \geq 3$, and D is a nonzero triangular derivation on $k[\mathbf{x}]$ with at most two dimensional support. Then, N_D is greater than or equal to $n - 3$. If $N_D = n - 3$, then, by a change of indices of the variables, we may write D as*

$$D = \kappa_0 \mathbf{x}^{\delta_0} \frac{\partial}{\partial x_{n-2}} + \kappa_1 \mathbf{x}^{\delta_1} x_{n-2}^{u_1 - 1} \frac{\partial}{\partial x_{n-1}} + \mathbf{x}^{\delta_2} x_{n-2}^{u_2 - 1} x_{n-1}^{v} \sum_{j=0}^{v} \kappa_{2,j} (\mathbf{x}^{\delta_1 - \delta_0} x_{n-2}^{u_1} x_{n-1}^{-1})^j \frac{\partial}{\partial x_n}. \quad (3.1)$$

Here, $\delta_j \in \mathbf{Z}_{\geq 0}{}^n$ for $j = 0, 1, 2$ whose last three components are zero, $u_1, u_2, v \in \mathbf{Z}$ with $u_1, u_2 \geq 1$ and $v \geq 0$, and $\kappa_0, \kappa_1, \kappa_{2,j} \in k$ for $j = 1, \ldots, v$ with $\kappa_0\kappa_1\kappa_{2,0} \neq 0$.

Proof. Note that, if $N_D = r$, then we may assume that $D(x_i) = 0$ for $i \leq n - r$ and $D(x_i) \neq 0$ for $i > n - r$ by a change of indices of the variables. We show this by induction on the number of indices $i \in \{1, \ldots, n\}$ with $D(x_i) \neq 0$ and $D(x_j) = 0$ for some $i < j \leq n$. Let $i \in \{1, \ldots, n\}$ be the maximum among such indices, and j the maximum index with $D(x_j) = 0$. Then, D remains triangular if we exchange i and j. By induction assumption, we may change indices so that $D(x_i) \neq 0$ implies that $D(x_j) \neq 0$ for any $j > i$.

Suppose that N_D was less than $n - 3$. Then, we may assume that $D(x_{n-3+i}) \neq 0$ for $i = 0, 1, 2, 3$, as mentioned above. Take $a_i \in \operatorname{supp}(x_{n-3+i}^{-1} D(x_{n-3+i}))$ for each i. Then, we have

$$\begin{pmatrix} a_1 - a_0 \\ a_2 - a_0 \\ a_3 - a_0 \end{pmatrix} = \begin{pmatrix} \ldots & -1 & 0 & 0 \\ \ldots\ldots\ldots & & -1 & 0 \\ \ldots\ldots\ldots\ldots & & & -1 \end{pmatrix}, \tag{3.2}$$

since D is triangular. Hence, $a_1 - a_0, a_2 - a_0, a_3 - a_0$ are linearly independent over $\mathbf{R}$. This contradicts that $\dim \operatorname{supp}(D) \leq 2$. Thus, $N_D = n - 3$.

Assume that $N_D = n - 3$. Then, we may assume that $D(x_{n-2+i}) \neq 0$ for $i = 0, 1, 2$. We show that D is written as (3.1). Take $a_i \in \operatorname{supp}(x_{n-2+i}^{-1} D(x_{n-2+i}))$ for $i = 0, 1, 2$. It suffices to show that $\operatorname{supp}(x_{n-2+i}^{-1} D(x_{n-2+i})) = \{a_i\}$ for $i = 0, 1$, and that $a_2 - a_2' \in \mathbf{Z}(a_1 - a_0)$ for every $a_2' \in \operatorname{supp}(x_n^{-1} D(x_n))$. First, suppose that there existed $a_0 \neq a_0' \in \operatorname{supp}(x_{n-2}^{-1} D(x_{n-2}))$. Then, since D is triangular, we have

$$\begin{pmatrix} a_0' - a_0 \\ a_1 - a_0 \\ a_2 - a_0 \end{pmatrix} = \begin{pmatrix} \cdots & 0 & 0 \\ \cdots & -1 & 0 \\ \ldots\ldots\ldots & & -1 \end{pmatrix}. \tag{3.3}$$

This shows that $a_0' - a_0, a_1 - a_0, a_2 - a_0$ are linearly independent. It contradicts that $\dim \operatorname{supp}(D) \leq 2$. Hence, $\operatorname{supp}(x_{n-2}^{-1} D(x_{n-2})) = \{a_0\}$. In a similar way, we see that $\operatorname{supp}(x_{n-1}^{-1} D(x_{n-1})) = \{a_1\}$.

Vectors $a_1 - a_0$ and $a_2 - a_0$ are linearly independent, since they are, respectively, the second and third rows of the right hand side of (3.3). Since the dimension of $\mathrm{supp}(D)$ is at most two, each $a_2' \in \mathrm{supp}(x_n^{-1}D(x_n))$ satisfies $a_2 - a_2' = \alpha(a_1 - a_0) + \beta(a_2 - a_0)$ for some $\alpha, \beta \in \mathbf{R}$. The n-th components of a_2 and a_2' are both equal to -1, while that of $a_2 - a_0$ is -1. Hence, $\beta = 0$. Namely, $a_2 - a_2' = \alpha(a_1 - a_0)$. Since the $(n-1)$-st component of $a_1 - a_0$ is -1, that of $a_2 - a_2'$ is equal to $-\alpha$. Thus, α is an integer. This completes the proof. $\square$

Let us denote $k[\mathbf{x}][\mathbf{y}] = k[\mathbf{x}][y_0, y_1, \dots, y_m]$ and $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] = k[\mathbf{x}, \mathbf{x}^{-1}][y_0, y_1, \dots, y_m]$, respectively, the polynomial rings in $m+1$ variables over $k[\mathbf{x}]$ and $k[\mathbf{x}, \mathbf{x}^{-1}]$. As before, we express monomials as $\mathbf{x}^a\mathbf{y}^b$ for $(a, b) \in \mathbf{Z}^n \times \mathbf{Z}^{m+1}$. For each $0 \neq f \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$, we set $\mathbf{e}(f)$ the unique element of $\mathbf{Z}^n$ such that

(a) $\mathbf{x}^{\mathbf{e}(f)} f \in k[\mathbf{x}][\mathbf{y}]$,

(b) $\mathbf{x}^a f \in k[\mathbf{x}][\mathbf{y}]$ implies that $a - \mathbf{e}(f) \in {\mathbf{Z}_{\geq 0}}^n$ for every $a \in \mathbf{Z}^n$.

Then, define $\rho(f) = \mathbf{x}^{\mathbf{e}(f)} f$.

In the situation of Lemma 3.1, we replace n by $n+3$ and $x_{n+1}, x_{n+2}, x_{n+3}$ by y_0, y_1, y_2, respectively. Then, the derivation (3.1) is described as a $k[\mathbf{x}]$-derivation

$$D = \kappa_0 \mathbf{x}^{\delta_0} \frac{\partial}{\partial y_0} + \kappa_1 \mathbf{x}^{\delta_1} y_0^{u_1 - 1} \frac{\partial}{\partial y_1} + \mathbf{x}^{\delta_2} y_0^{u_2 - 1} y_1^v \sum_{j=0}^{v} \kappa_{2,j} (\mathbf{x}^{\delta_1 - \delta_0} y_0^{u_1} y_1^{-1})^j \frac{\partial}{\partial y_2} \tag{3.4}$$

on $k[\mathbf{x}][\mathbf{y}]$ for $m = 2$, where $\delta_j \in {\mathbf{Z}_{\geq 0}}^n$ for $j = 0, 1, 2$, $u_1, u_2, v \in \mathbf{Z}$ with $u_1, u_2 \geq 1$ and $v \geq 0$, and $\kappa_0, \kappa_1, \kappa_{2,j} \in k$ for $j = 1, \dots, v$ with $\kappa_0, \kappa_1, \kappa_{2,0} \neq 0$.

We set $\epsilon_{i,j} = \delta_i - \delta_j$ for i, j. Then, define two elements in $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ by

$$\tilde{F} = y_1 - \frac{\kappa_1}{\kappa_0 u_1} \mathbf{x}^{\epsilon_{1,0}} y_0^{u_1}, \tag{3.5}$$

and

$$\tilde{G} = y_2 - \sum_{p=0}^{v} \left(\sum_{q=0}^{p} \phi(p, q) \right) \mathbf{x}^{p\epsilon_{1,0} + \epsilon_{2,0}} y_0^{pu_1 + u_2} y_1^{v-p}, \tag{3.6}$$

where $\phi(p, q)$ is an element of k defined by

$$\phi(p, q) = \frac{(-\kappa_1)^{p-q} \kappa_{2,q}}{\kappa_0^{p-q+1}(v - q + 1)} \prod_{r=1}^{p-q} \frac{v - q - r + 1}{(q + r)u_1 + u_2} \tag{3.7}$$

for p, q. Then, it follows that $D(\tilde{F}) = D(\tilde{G}) = 0$. It is easily checked that $D(\tilde{G}) = 0$. We verify the equality $D(\tilde{G}) = 0$ only.

We set

$$P(p,q) = \left(\frac{(-\kappa_1)^{p-q}\kappa_{2,q}(v-p+1)}{\kappa_0^{p-q}(v-q+1)} \prod_{r=1}^{p-q-1} \frac{v-q-r+1}{(q+r)u_1+u_2} \right) \mathbf{x}^{p\epsilon_{1,0}+\delta_2} y_0^{pu_1+u_2-1} y_1^{v-p} \quad (3.8)$$

for p, q. Then, $D(y_2) = \sum_{p=0}^{v} P(p,p)$,

$$D(\phi(p,q)\mathbf{x}^{p\epsilon_{1,0}+\epsilon_{2,0}} y_0^{pu_1+u_2} y_1^{v-p}) = P(p,q) - P(p+1,q) \quad (3.9)$$

for $0 \leq q \leq p \leq v$, and $P(v+1,q) = 0$ for $q = 0, \ldots, p$. Hence, we have

$$D(\tilde{G}) = \sum_{p=0}^{v} P(p,p) - \sum_{p=0}^{v}\sum_{q=0}^{p}(P(p,q) - P(p+1,q)) \quad (3.10)$$

$$= \sum_{q=0}^{v}\left(P(q,q) - \sum_{p=q}^{v}(P(p,q) - P(p+1,q)) \right) = 0. \quad (3.11)$$

We set $\xi = \sum_{q=0}^{v} \phi(v,q)$. Put $u_i' = u_i/\gcd(u_1,u_2)$ for $i = 1, 2$, and set $\eta = u_1'\epsilon_{2,0} - u_2'\epsilon_{1,0}$, $w = u_1'v + u_2'$. If $\xi \neq 0$, then set

$$\tilde{H} = \mathbf{x}^{\eta}\tilde{F}^{w} - (-1)^{w+u_1'} \frac{\kappa_1^{w}}{(\kappa_0 u_1)^{w}\xi^{u_1'}} \tilde{G}^{u_1'}. \quad (3.12)$$

We define $F = \rho(\tilde{F})$ and $G = \rho(\tilde{G})$. If $\xi \neq 0$, then define $H = \rho(\tilde{H})$, else set $H = 0$.

Then, we have the following.

Theorem 3.2 *Assume that $m = 2$, and D is a $k[\mathbf{x}]$-derivation on $k[\mathbf{x}][\mathbf{y}]$ which has the form of* (3.4)*. Then, $\{x_1, \ldots, x_n, F, G, H\}$ is a universal SAGBI basis for the kernel $k[\mathbf{x}][\mathbf{y}]^D$ of D. In particular, $k[\mathbf{x}][\mathbf{y}]^D$ is generated by at most $n+3$ elements over k.*

Before proving Theorem 3.2, we recall a fact on the kernel of a locally nilpotent derivation. Let B be a k-algebra, and D a locally nilpotent derivation on B. An element $s \in B$ is said to be a *slice* of D if $D(s) = 1$. Assume that D has a slice s in B. Then, for each $b \in B$, we define

$$\Psi_s(b) = \sum_{p=0}^{\infty} \frac{(-s)^p}{p!} D^p(b). \quad (3.13)$$

Note that every summand of (3.13) is zero for sufficiently large p, since D is locally nilpotent. By definition, it follows that $\Psi_s(s) = 0$, and $\Psi_s(b) = b$ for any $a \in B^D$. The following fact is well-known (See [1, Corollary 1.3.23] for instance).

Lemma 3.3 *The map $B \ni b \mapsto \Psi_s(b) \in B$ is a homomorphism of k-algebras. Its image $\Psi_s(B)$ is equal to the kernel B^D. In particular, if $S \subset B$ generates B over k, then $\{\Psi_s(b) \mid b \in S\}$ generates B^D over k.*

Assume that D is a triangular derivation on $k[\mathbf{x}]$. If $D(x_1) \neq 0$, then $s = x_1/D(x_1)$ is a slice of D. The following is a consequence of Lemma 3.3.

Corollary 3.4 *Assume that D is a triangular derivation on $k[\mathbf{x}]$ with $D(x_1) \neq 0$. We set $s = x_1/D(x_1)$. Then, $\{\Psi_s(x_2), \ldots, \Psi_s(x_n)\}$ is a SAGBI basis for $k[\mathbf{x}]^D$ for any monomial order $\preceq$ on $k[\mathbf{x}]$ with $x_i = \mathrm{in}_{\preceq}(\Psi_s(x_i))$ for $i = 2, \ldots, n$.*

Note that such a monomial order $\preceq$ on $k[\mathbf{x}]$ as in Corollary 3.4 always exists. For example, assume that $\preceq$ is the monomial order defined by $a \preceq b$ if the last nonzero component of $b - a$ is positive for $a, b \in \mathbf{Z}^n$. Then, $x_i = \mathrm{in}_{\preceq}(\Psi_s(x_i))$ for $i = 2, \ldots, n$.

Proof. Since D is triangular, it is locally nilpotent. By Lemma 3.3, $\{\Psi_s(x_2), \ldots, \Psi_s(x_n)\}$ generates $k[\mathbf{x}]^D$ over k, since $\Psi_s(x_1) = 0$.

By the choice of $\preceq$, we have

$$\mathrm{in}_{\preceq}(k[\mathbf{x}]^D) \supset k[x_2, \ldots, x_n]. \tag{3.14}$$

On the other hand, it follows that $\mathrm{trans.deg}_k\, A \geq \mathrm{trans.deg}_k\, \mathrm{in}_{\preceq}(A)$ for a k-subalgebra $A \subset k[\mathbf{x}]$ by the following reason. Take $f_1, \ldots, f_r \in A$ so that their initial terms are a transcendence basis of $\mathrm{in}_{\preceq}(A)$ over k. Suppose that

$$\sum_{(i_1,\ldots,i_r) \in \mathbf{Z}_{\geq 0}{}^r} \alpha_{i_1,\ldots,i_r} f_1^{i_1} \cdots f_r^{i_r} = 0 \tag{3.15}$$

for some $\alpha_{i_1,\ldots,i_r} \in k$ which is not zero for some $(i_1, \ldots, i_r)$. Let $(i_1, \ldots, i_r)$ be an element of $\mathbf{Z}_{\geq 0}{}^r$ with $\alpha_{i_1,\ldots,i_r} \neq 0$ such that $\mathrm{in}_{\preceq}(f_1^{i'_1}, \cdots f_r^{i'_r}) \preceq \mathrm{in}_{\preceq}(f_1^{i_1}, \cdots f_r^{i_r})$ for any

$(i'_1, \ldots, i'_r)$ with $\alpha_{i'_1,\ldots,i'_r} \neq 0$. Then, there exists $(i_1, \ldots, i_r) \neq (j_1, \ldots, j_r) \in \mathbf{Z}_{\geq 0}{}^r$ with $\alpha_{j_1,\ldots,j_r} \neq 0$ such that $\mathrm{in}_{\preceq}(f_1^{i_1} \cdots f_r^{i_r})$ is equal to $\mathrm{in}_{\preceq}(f_1^{j_1} \cdots f_r^{j_r})$ up to scalar multiplication. Actually, if such $(j_1, \ldots, j_r)$ did not exist, then the initial term of the left of (3.15) would be the nonzero term $\alpha_{i_1,\ldots,i_r} \mathrm{in}_{\preceq}(f_1^{i_1} \cdots f_r^{i_r})$. Since $\mathrm{in}_{\preceq}(f_1^{i_1} \cdots f_r^{i_r}) = \prod_{l=1}^{r} \mathrm{in}_{\preceq}(f_l)^{i_l}$ and $\mathrm{in}_{\preceq}(f_1^{j_1} \cdots f_r^{j_r}) = \prod_{l=1}^{r} \mathrm{in}_{\preceq}(f_l)^{j_l}$ by (1.3), it implies the algebraic dependence of $\mathrm{in}_{\preceq}(f_1), \ldots, \mathrm{in}_{\preceq}(f_r)$ over k. This is a contradiction. Thus, $\mathrm{trans.deg}_k A \geq \mathrm{trans.deg}_k \mathrm{in}_{\preceq}(A)$.

Since k is of characteristic zero, the transcendence degree of $k[\mathbf{x}]^D$ is less than n. Hence, that of $\mathrm{in}_{\preceq}(k[\mathbf{x}]^D)$ is less than n. So, (3.14) implies that $\mathrm{in}_{\preceq}(k[\mathbf{x}]^D) = k[x_2, \ldots, x_n]$. Therefore, $\{\Psi_s(x_2), \ldots, \Psi_s(x_n)\}$ is a SAGBI basis for $k[\mathbf{x}]^D$ with respect to $\preceq$. $\square$

Assume that D is a triangular derivation on $k[\mathbf{x}]$ with $N_D = n-2$. Then, there exist $1 \leq p < q \leq n$ such that $D(x_p), D(x_q) \neq 0$ and $D(x_i) = 0$ for any $i \neq p, q$. We set $s = x_p/D(x_p)$. Then, D is extended uniquely to a locally nilpotent derivation on $k[\mathbf{x}][s]$. Write $\Psi_s(x_q) = h/h'$, where $h, h' \in k[\mathbf{x}]$ with $\gcd(h_0, h_1) = 1$.

Corollary 3.5 *Assume that D is a triangular derivation on $k[\mathbf{x}]$. If there exist $1 \leq p < q \leq n$ such that $D(x_p), D(x_q) \neq 0$ and $D(x_i) = 0$ for any $i \neq p, q$, then*

$$\{x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_n, h\} \tag{3.16}$$

is a universal SAGBI basis for $k[\mathbf{x}]^D$.

Proof. We set $k[\mathbf{x}'] = k[x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_n]$. Then, $k[\mathbf{x}'] \subset k[\mathbf{x}]^D$. We show that $h \in k[\mathbf{x}]^D$. Since D is triangular, $D(x_p) \in k[x_1, \ldots, x_{p-1}]$. Hence, the denominator of each irreducible fraction in $k[\mathbf{x}][s]$ is contained in $k[x_1, \ldots, x_{p-1}]$. So, h' is in $k[x_1, \ldots, x_{p-1}]$. Since $D(h/h') = 0$, this implies that $h \in k[\mathbf{x}]^D$. Thus, we have $k[\mathbf{x}'][h] \subset k[\mathbf{x}]^D$.

Suppose that there existed $f \in k[\mathbf{x}]^D \setminus k[\mathbf{x}'][h]$. Since $\Psi_s(x_p) = \Psi_s(s) = 0$, we have

$$k[\mathbf{x}]^D = k[\mathbf{x}][s]^D \cap k[\mathbf{x}] = \Psi_s(k[\mathbf{x}][s]) \cap k[\mathbf{x}] = k[\mathbf{x}'][h/h'] \cap k[\mathbf{x}] \tag{3.17}$$

by Lemma 3.3. Hence, we may write $f = f_0(h/h')^r + f_1(h/h')^{r-1} + \cdots + f_r$, where $f_i \in k[\mathbf{x}'][h]$ for each i. Assume that r is the minimum among such expressions. Then, $r > 0$. Moreover, h' does not divide f_0. Actually, if h' divided f_0, then we may express $f = (f_0h/h' + f_1)(h/h')^{r-1} + \cdots + f_r$. Since $f_0h/h' + f_1 \in k[\mathbf{x}'][h]$, it contradicts the minimality of r. Thus, h' does not divide f_0. We set $f' = f_0h^r + f_1h^{r-1}h' \cdots + f_r(h')^r$. Then, $f = f'/(h')^r$. However, f' is not divided by h'. This contradicts that $f \in k[\mathbf{x}]$. Therefore, we have $k[\mathbf{x}]^D = k[\mathbf{x}'][h]$.

Let $\preceq$ be any monomial order on $k[\mathbf{x}]$. We show that $\mathrm{in}_\preceq(k[\mathbf{x}]^D) = k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Assume that $f \in k[\mathbf{x}]^D$. Then, $f = f_0h^r + \cdots + f_r$ for some r and $f_i \in k[\mathbf{x}']$. Note that each monomial of h is divided by x_p or x_q. We set a_i the power of x_i in $\mathrm{in}_\preceq(h)$ for $i = p, q$. Then, for j with $f_j \neq 0$, the i-th component of $v_\preceq(f_jh^{r-j})$ is $(r-j)a_i$ for $i = p, q$. Hence, $\mathrm{in}_\preceq(f_ih^{r-i}) \neq \mathrm{in}_\preceq(f_jh^{r-j})$ for any $i \neq j$ with $f_i, f_j \neq 0$. This implies that $\mathrm{in}_\preceq(f) = \mathrm{in}_\preceq(f_ih^{r-i})$ for some i. Since $\mathrm{in}_\preceq(f_ih^{r-i}) \in k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$, we have $\mathrm{in}_\preceq(f) \in k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Thus, $\mathrm{in}_\preceq(k[\mathbf{x}]^D) \subset k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Since the converse inclusion is clear, we get $\mathrm{in}_\preceq(k[\mathbf{x}]^D) = k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Therefore, (3.16) is a universal SAGBI basis for $k[\mathbf{x}]^D$. □

We will show Theorem 3.2, as a consequence of Theorem 3.6 below. Let M be a submodule of $\mathbf{Z}^n \times \mathbf{Z}^{m+1}$ of rank two which is not contained in

$$L = \{(a, (b_0, b_1, \ldots, b_m)) \in \mathbf{Z}^n \times \mathbf{Z}^{m+1} \mid b_0 = 0\}. \tag{3.18}$$

Let $\Psi : k[\mathbf{x}][y_1, \ldots, y_m] \to k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ be a homomorphism of $k[\mathbf{x}]$-algebras such that

$$\Psi(y_i) - y_i \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0 \quad \text{and} \quad \mathrm{supp}(y_i^{-1}\Psi(y_i)) \subset M \tag{3.19}$$

for $i = 1, \ldots, m$. Let $\Phi : k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \to k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ be the homomorphism defined by substituting zero for y_0. Then, consider the k-subalgebra

$$A = \Psi(k[\mathbf{x}][y_1, \ldots, y_m]) \cap k[\mathbf{x}][\mathbf{y}] \tag{3.20}$$

of $k[\mathbf{x}][\mathbf{y}]$. Put $F_i = \rho(\Psi(y_i))$ for $i = 1, \ldots, m$. Let $\bar{\eta} = (\bar{\eta}', \bar{\eta}'') \in \mathbf{Z}^n \times \mathbf{Z}^{m+1}$ such that $M \cap L = \mathbf{Z}\bar{\eta}$. We set $\bar{\eta}_1''$ the vector $\bar{\eta}''$ whose negative components are replaced by zero, and $\bar{\eta}''_2 = \bar{\eta}''_1 - \bar{\eta}''$. Then, define $\tilde{H}(\beta) = \mathbf{x}^{\bar{\eta}'}\Psi(\mathbf{y}^{\bar{\eta}''_1}) - \beta\Psi(\mathbf{y}^{\bar{\eta}''_2})$ and $H(\beta) = \rho(\tilde{H}(\beta))$ for each $\beta \in \bar{k}$. We define the *Newton polytope* $\mathrm{New}(f)$ of f by the convex hull of $\mathrm{supp}(f)$ in $\mathbf{R}^n$ for each $f \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$. Then, there exist finite number of elements $\mu_0, \mu_1, \ldots, \mu_\nu \in k \setminus \{0\}$ such that

(i) $\mathrm{New}(\tilde{H}(\mu_i)) \neq \mathrm{New}(\tilde{H}(\mu_j))$ if $i \neq j$.

(ii) $\mathrm{New}(\tilde{H}(\mu_0))$ contains $\mathrm{supp}(\mathbf{x}^{\bar{\eta}'}\Psi(\mathbf{y}^{\bar{\eta}''_1}))$ and $\mathrm{supp}(\Psi(\mathbf{y}^{-\bar{\eta}''_2}))$.

(iii) $\mathrm{New}(\tilde{H}(\beta)) = \mathrm{New}(\tilde{H}(\mu_0))$ for all $\beta \in \bar{k} \setminus \{0, \mu_1, \ldots, \mu_\nu\}$.

With the notation above, we have the following.

Theorem 3.6 *The set* $\{x_1, \ldots, x_n, F_1, \ldots, F_m, H(\mu_1), \ldots, H(\mu_\nu)\}$ *is a universal SAGBI basis for* A.

Proof. We set $\Gamma = (\mathbf{Z}^m \times \mathbf{Z}^3)/M$, and define $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ the k-vector space generated by monomials $\mathbf{x}^a\mathbf{y}^b \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ with the image of (a, b) in Γ is equal to γ for each γ. Then, the Γ-grading structure $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ is defined. For this Γ-grading, Ψ and Φ send each Γ-homogeneous element to that of the same Γ-degree. Note that $\Psi(\Phi(f)) = f$ for $f \in \Psi(k[\mathbf{x}][y_1, \ldots, y_m])$. We show that $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where $A_\gamma = A \cap k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for $\gamma \in \Gamma$. Each $f \in A$ is written as $f = \sum_\gamma f_\gamma$, where $f_\gamma \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$. Since the supports of f_γ and $f_{\gamma'}$ do not intersect if $\gamma \neq \gamma'$, f_γ is in $k[\mathbf{x}][\mathbf{y}]$ for any γ. It follows that $f = \Psi(\Phi(f)) = \sum_{\gamma\in\Gamma} \Psi(\Phi(f_\gamma))$. Since Φ and Ψ preserve Γ-degree, $\Psi(\Phi(f_\gamma))$ is in $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for each γ. Hence, $f_\gamma = \Psi(\Phi(f_\gamma))$. Thus, $f_\gamma \in A_\gamma$ for each $\gamma \in \Gamma$. Therefore, $A \subset \bigoplus_{\gamma\in\Gamma} A_\gamma$. The converse inclusion is clear.

Take any $\preceq \in \Omega$, and set S the subsemigroup of $\mathbf{Z}^n \times \mathbf{Z}^{m+1}$ generated by ${\mathbf{Z}_{\geq 0}}^n \times \{0\}$, $v_\preceq(F_i)$ for $i = 1, \ldots, m$ and $v_\preceq(H(\mu_i))$ for $i = 1, \ldots, \nu$. To complete the proof, it suffices to show that $v_\preceq(f) \in S$ for any Γ-homogeneous element $f \in A \setminus \{0\}$. First, we show that $v_\preceq(H(\mu)) \in S$ for any $\mu \in \bar{k} \setminus \{0\}$. It is true if $\mu = \mu_i$ for $i = 1, \ldots, \nu$. For $\mu \in \bar{k} \setminus \{0, \mu_1, \ldots, \mu_\nu\}$, it follows that $\mathrm{New}(H(\mu)) = \mathrm{New}(H(\mu_0))$ by the condition (iii). Hence, $v_\preceq(H(\mu)) = v_\preceq(H(\mu_0))$ for $\mu \in \bar{k} \setminus \{0, \mu_1, \ldots, \mu_\nu\}$. So, it suffices to show that $v_\preceq(H(\mu_0)) \in S$. By the condition (ii), $v_\preceq(H(\beta_0)) = v_\preceq(h_j)$ for some $j \in \{1, 2\}$, where

$h_1 = \mathbf{x}^{\bar{\eta}'}\Psi(\mathbf{y}^{\bar{\eta}''_1})$ and $h_2 = \Psi(\mathbf{y}^{\bar{\eta}''_2})$. Since $H(\beta_0) = \mathbf{x}^{\mathbf{e}(\tilde{H}(\beta_0))}\tilde{H}(\beta_0)$, we have

$$v_{\preceq}(H(\beta_0)) = \big(\mathbf{e}(\tilde{H}(\beta_0)), 0\big) + v_{\preceq}(h_j) = \big(\mathbf{e}(\tilde{H}(\beta_0)) - \mathbf{e}(h_j), 0\big) + v_{\preceq}(\rho(h_j)). \tag{3.21}$$

The condition (ii) also implies that $\mathbf{e}(\tilde{H}(\beta_j)) - \mathbf{e}(h_i) \in {\mathbf{Z}_{\geq 0}}^n$. Since $v_{\preceq}(\rho(h_j))$ is contained in $\sum_{i=1}^m \mathbf{Z}_{\geq 0} v_{\preceq}(F_i)$, we have $v_{\preceq}(\rho(h_j)) \in S$. Thus, $v_{\preceq}(H(\beta_0))$ is in S. Therefore, $v_{\preceq}(H(\mu))$ is in S for any $\mu \in \bar{k} \setminus \{0\}$.

Now, let f be a Γ-homogeneous element of $A \setminus \{0\}$. Then, there exist $a \in \mathbf{Z}^n$, $b_1, \ldots, b_m, r \in \mathbf{Z}_{\geq 0}$ and $\alpha_i \in k$ with $\alpha_0, \alpha_r \neq 0$ such that

$$\Phi(f) = \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \mathbf{y}^{r\bar{\eta}''_2} \sum_{i=0}^{r} \alpha_i (\mathbf{x}^{\bar{\eta}'}\mathbf{y}^{\bar{\eta}''_1}\mathbf{y}^{-\bar{\eta}''_2})^i. \tag{3.22}$$

Actually, we have $\Phi(f) = \mathbf{x}^a\mathbf{y}^{b'} \sum_{i=0}^r \alpha_i(\mathbf{x}^{\bar{\eta}'}\mathbf{y}^{\bar{\eta}''_1 - \bar{\eta}''_2})^i$ for some $a \in \mathbf{Z}^n$, $b' \in \mathbf{Z}^{m+1}$, $r \in \mathbf{Z}_{\geq 0}$ and $\alpha_i \in k$ with $\alpha_0, \alpha_r \neq 0$, since every $c, d \in \operatorname{supp}(\Phi(f))$ satisfy $c - d \in \mathbf{Z}\bar{\eta}$. The condition $\Phi(f) \in k[\mathbf{x}][y_1, \ldots, y_m]$ implies that the first component of b' is zero and $b' - r\bar{\eta}''_2 \in {\mathbf{Z}_{\geq 0}}^{m+1}$. Hence, we get the expression (3.22). Let $\beta_1, \ldots, \beta_r \in \bar{k}$ be the solutions of the equation $\sum_{i=0}^r \alpha_i X^i = 0$. Then, $\beta_i \neq 0$ for any i. We may write (3.22) as

$$\begin{aligned}\Phi(f) &= \alpha_0 \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \mathbf{y}^{r\bar{\eta}''_2} \prod_{i=1}^r (\mathbf{x}^{\bar{\eta}'}\mathbf{y}^{\bar{\eta}''_1 - \bar{\eta}''_2} - \beta_i) && (3.23)\\ &= \alpha_0 \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \prod_{i=1}^r (\mathbf{x}^{\bar{\eta}'}\mathbf{y}^{\bar{\eta}''_1} - \beta_i \mathbf{y}^{\bar{\eta}''_2}). && (3.24)\end{aligned}$$

Since $f = \Psi(\Phi(f))$, we get

$$\begin{aligned}f &= \alpha_0 \mathbf{x}^a \Psi(y_1)^{b_1} \cdots \Psi(y_m)^{b_m} \prod_{i=1}^r (\mathbf{x}^{\bar{\eta}'}\Psi(\mathbf{y}^{\bar{\eta}''_1}) - \beta_i \Psi(\mathbf{y}^{\bar{\eta}''_2})) && (3.25)\\ &= \alpha_0 \mathbf{x}^{a'} \left(\prod_{j=1}^m F_j^{b_j}\right)\left(\prod_{i=1}^r H(\beta_i)\right), && (3.26)\end{aligned}$$

where $a' = a - \sum_{j=1}^m b_j \mathbf{e}(\Psi(y_j)) - \sum_{i=1}^r \mathbf{e}(\tilde{H}(\beta_i))$. By (3.26), we have

$$v_{\preceq}(f) = (a', 0) + \sum_{j=1}^m b_j v_{\preceq}(F_j) + \sum_{i=1}^r v_{\preceq}(H(\beta_i)). \tag{3.27}$$

Clearly, $\sum_{j=1}^{m} b_j v_{\preceq}(F_j)$ is in S. As we showed above, $v_{\preceq}(H(\beta_i)) \in S$ for each i. We show that $(a', 0) \in S$. Suppose the contrary. Then, the j-th component of a' is negative for some j. Since $f \in k[\mathbf{x}][\mathbf{y}]$, the polynomial $(\prod_{j=1}^{m} F_j^{b_j})(\prod_{i=1}^{r} H(\beta_i))$ is divided by x_j. However, x_j does not divide $\rho(g)$ for any $g \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \setminus \{0\}$ by definition. This is a contradiction. Hence, $(a', 0) \in S$. Therefore, $v_{\preceq}(f)$ is in S. This completes the proof. $\square$

To prove Theorem 3.2, we need the following two lemmas. Let D be the k-derivation (3.4), and put $s = y_0/\kappa_0 \mathbf{x}^{\delta_0}$. Then, D is naturally extended to a locally nilpotent derivation on $k[\mathbf{x}][\mathbf{y}][s]$. Note that s is a slice of this derivation. We set M the submodule of $\mathbf{Z}^n \times \mathbf{Z}^3$ generated by $\big(\epsilon_{1,0}, (u_1, -1, 0)\big)$ and $\big(\epsilon_{2,0}, (u_2, v, -1)\big)$. Then, it follows that $M \cap L = \mathbf{Z}\big(\eta, (0, w, -u_1')\big)$.

Lemma 3.7 (i) $k[\mathbf{x}][\mathbf{y}]^D = \Psi_s(k[\mathbf{x}][y_1, y_2]) \cap k[\mathbf{x}][\mathbf{y}]$.
(ii) *The map* $k[\mathbf{x}][y_1, y_2] \ni f \mapsto \Psi_s(f) \in k[\mathbf{x}][\mathbf{y}][s]^D$ *is an isomorphism. Its inverse is* $k[\mathbf{x}][\mathbf{y}][s]^D \ni f \mapsto \Phi(f) \in k[\mathbf{x}][y_1, y_2]$.
(iii) $\Psi_s(y_1) = \tilde{F}$ *and* $\Psi_s(y_1) = \tilde{G}$.
(iv) $\Psi_s(y_i) - y_i \subset k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0$ *and* $\operatorname{supp}(y_i^{-1}\Psi_s(y_i)) \subset M$ *for* $i = 1, 2$.

Proof. By Lemma 3.3, we get $k[\mathbf{x}][\mathbf{y}][s]^D = \Psi_s(k[\mathbf{x}][\mathbf{y}][s])$. Since $\Psi_s(y_0) = \Psi_s(s) = 0$, it is equal to $\Psi_s(k[\mathbf{x}][y_1, y_2])$. Therefore,

$$k[\mathbf{x}][\mathbf{y}]^D = k[\mathbf{x}][\mathbf{y}][s]^D \cap k[\mathbf{x}][\mathbf{y}] = \Psi_s(k[\mathbf{x}][y_1, y_2]) \cap k[\mathbf{x}][\mathbf{y}]. \tag{3.28}$$

Hence, (i) is proved.

For $f \in k[\mathbf{x}][y_1, y_2]$, we have $\Psi_s(f) = f - s\sum_{p=1}^{\infty}(-s)^{p-1}D^p(f)/p!$. Hence, $\Phi(\Psi_s(f)) = f$. Moreover, $\Psi_s(k[\mathbf{x}][y_0, y_1]) = k[\mathbf{x}][\mathbf{y}][s]^D$ as above. This proves (ii).

Note that $\tilde{F}, \tilde{G} \in k[\mathbf{x}][s]^D$. Since $\Phi(\tilde{F}) = y_1$ and $\Phi(\tilde{G}) = y_2$, we have $\Psi_s(y_1) = \tilde{F}$ and $\Psi_s(y_2) = \tilde{G}$ by (ii). Hence, we get (iii).

(iv) is a consequence of (iii). Actually, we see easily that $\tilde{F} - y_1, \tilde{G} - y_2 \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0$ and $\operatorname{supp}(y_1^{-1}\tilde{F}), \operatorname{supp}(y_2^{-1}\tilde{G}) \subset M$. This completes the proof. $\square$

We set $\bar{\eta}_1'' = w$ and $\bar{\eta}_2'' = u_1'$. Then, define $\tilde{H}(\beta) = \mathbf{x}^\eta \Psi_s(y_1)^{\bar{\eta}_1''} - \beta \Psi_s(y_2)^{\bar{\eta}_2''}$ and $H(\beta) = \rho(\tilde{H}(\beta))$ for each $\beta \in \bar{k} \setminus \{0\}$. If $\xi \neq 0$, then put

$$\mu_1 = (-1)^{w+u_1'} \frac{\kappa_1^w}{(\kappa_0 u_1)^w \xi^{u_1'}}, \tag{3.29}$$

and take any $\mu_0 \in k \setminus \{0, \mu_1\}$. If $\xi = 0$, then take any $\mu_0 \in k \setminus \{0\}$.

Lemma 3.8 *Assume that* $\xi \neq 0$. *Then, we have*

(i) $\mathrm{New}(\tilde{H}(\mu_0)) \neq \mathrm{New}(\tilde{H}(\mu_1))$.

(ii) $\mathrm{New}(\tilde{H}(\mu_0))$ *contains* $\mathrm{supp}(\mathbf{x}^{\bar{\eta}'} \Psi(y_1)^{\bar{\eta}_1''})$ *and* $\mathrm{supp}(\Psi(y_2)^{\bar{\eta}_2''})$.

(iii) $\mathrm{New}(\tilde{H}(\beta)) = \mathrm{New}(\tilde{H}(\mu_0))$ *for all* $\beta \in \bar{k} \setminus \{0, \mu_1\}$.

Assume that $\xi = 0$. *Then, we have*

(iv) $\mathrm{New}(\tilde{H}(\mu_0))$ *contains* $\mathrm{supp}(\mathbf{x}^{\bar{\eta}'} \Psi(y_1)^{\bar{\eta}_1''})$ *and* $\mathrm{supp}(\Psi(y_2)^{\bar{\eta}_2''})$.

(v) $\mathrm{New}(\tilde{H}(\beta)) = \mathrm{New}(\tilde{H}(\mu_0))$ *for all* $\beta \in \bar{k} \setminus \{0\}$.

Proof. Note that $\mathbf{x}^\eta \Psi_s(y_1)^{\bar{\eta}_1''} = \mathbf{x}^\eta \tilde{F}^w$ and $\Psi_s(y_2)^{\bar{\eta}_2''} = \tilde{G}^{u_1'}$ by Lemma 3.7 (iii). Assume that $\xi \neq 0$. Then, the sets of the vertices of $\mathrm{New}(\mathbf{x}^\eta \Psi_s(y_1)^{\bar{\eta}_1''})$ and $\mathrm{New}(\Psi_s(y_2)^{\bar{\eta}_2''})$ are, respectively, $\{a_1, a_2\}$, and $\{b_1, b_2, b_3\}$. Here,

$$\begin{aligned} a_1 &= \big(\eta, (0, w, 0)\big) & b_1 &= \big(0, (0, 0, u_1')\big) \\ a_2 &= \big(\eta + w\epsilon_{1,0}, (u_1 w, 0, 0)\big) & b_2 &= \big(u_1'(\epsilon_{2,0} + v\epsilon_{1,0}), (u_1'(u_1 v + u_2), 0, 0)\big) \\ & & b_3 &= \big(u_1'\epsilon_{2,0}, (u_1' u_2, u_1' v, 0)\big). \end{aligned}$$

We show that $a_1, a_2, b_1, b_2, b_3 \in \mathrm{New}(\tilde{H}(\beta))$ for any $\beta \in \bar{k} \setminus \{0, \mu_1\}$ and $a_2 \notin \mathrm{New}(\tilde{H}(\mu_1))$. The assertions (i), (ii), (iii) follow from this. Take any $\beta \in \bar{k} \setminus \{0, \mu_1\}$. Since $a_1 \notin \mathrm{New}(\Psi_s(y_2)^{\bar{\eta}_2''})$ and $b_1 \notin \mathrm{New}(\mathbf{x}^\eta \Psi_s(y_1)^{\bar{\eta}_1''})$, we have $a_1, b_1 \in \mathrm{New}(\tilde{H}(\beta))$. It follows that

$$\eta + w\epsilon_{1,0} = (u_1'\epsilon_{2,0} - u_2'\epsilon_{1,0}) + (u_1' v + u_2')\epsilon_{1,0} = u_1'(\epsilon_{2,0} + v\epsilon_{1,0}), \tag{3.30}$$

and $u_1 w = u_1'(u_1 v + u_2)$. Hence, $a_2 = b_2$. We remark that the coefficient of $\mathbf{x}^{\eta + w\epsilon_{1,0}} y_0^{u_1 w}$ in $\tilde{H}(\beta')$ is zero if and only if $\beta' = \mu_1$ for $\beta' \in \bar{k}$. Hence, a_2 is in $\mathrm{New}(\tilde{H}(\beta))$. Since $b_3 = (1 - u_2'/w)a_1 + (u_2'/w)a_2$, we have $b_3 \in \mathrm{New}(\tilde{H}(\beta))$. Therefore, a_1, a_2, b_1, b_2, b_3 are contained in $\mathrm{New}(\tilde{H}(\beta))$ for any $\beta \in \bar{k} \setminus \{0, \mu_1\}$. The first component of the second factor of a_2 is greater than that of any element of $\{a_1, b_1, b_3\} \backslash \{a_2\}$. Hence, it is greater than that of any element of $\mathrm{supp}(\tilde{H}(\mu_1))$ but a_2. Since $a_2 \notin \mathrm{supp}(\tilde{H}(\mu_1))$, we get $a_2 \notin \mathrm{New}(\tilde{H}(\mu_1))$. This proves the lemma for $\xi \neq 0$.

Assume that $\xi = 0$. In a similar way as above, a_1, b_1 are in $\mathrm{New}(\tilde{H}(\beta))$. Since the coefficient of $\mathbf{x}^{\eta + w\epsilon_{1,0}} y_0^{u_1 w}$ in $\tilde{H}(\beta)$ is equal to that in $\mathbf{x}^{\eta} \Psi_s(y_1)^{\bar{\eta}_1''}$, we have $a_2 \in \mathrm{supp}(\tilde{H}(\beta))$. Then, b_3 is in $\mathrm{New}(\tilde{H}(\beta))$ similarly. Thus, $a_1, a_2, b_1, b_3 \in \mathrm{New}(\tilde{H}(\beta))$. This implies (iv) and (v). Therefore, the lemma is proved. $\square$

We complete the proof of Theorem 3.2. Assume that $\xi \neq 0$. Then, by Theorem 3.6, Lemma 3.7 (i), (iv), and Lemma 3.8, the set $\{x_1, \ldots, x_n, \rho(\Psi_s(y_1)), \rho(\Psi_s(y_2)), H(\mu_1)\}$ is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$. Since $\Psi_s(y_1) = \tilde{F}$ and $\Psi_s(y_2) = \tilde{G}$ by Lemma 3.7 (iii), we have $\rho(\Psi_s(y_1)) = F$ and $\rho(\Psi_s(y_2)) = G$. Moreover, $H(\mu_1) = H$ by definition. Thus, the theorem is proved if $\xi \neq 0$. Similarly, $\{x_1, \ldots, x_l, F, G\}$ is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$ if $\xi = 0$. Therefore, the proof of Theorem 3.2 is completed.

Now, Theorem 1.2 is a consequence of the results above. Actually, it follows from what we mentioned above Lemma 3.1, Corollary 3.5 and Theorem 3.2. In each case, we determined the universal SAGBI basis explicitly.

In [7], Maubach showed that, for $n = 4$, the kernel of a triangular derivation D on $k[\mathbf{x}]$ is generated by at most four elements if $D(x_i)$ is a monomial multiplied by an element of k for each i. Furthermore, we know a SAGBI basis for this kernel.

Corollary 3.9 *Assume that $n = 4$. Let D be a triangular derivation on $k[\mathbf{x}]$ such that $x_i^{-1} D(x_i) = \kappa_i \mathbf{x}^{d_i}$ for some $\kappa_i \in k$ and $d_i \in \mathbf{Z}^4$ for each i. Then, $k[\mathbf{x}]^D$ has a universal SAGBI basis with at most four elements if $\kappa_i = 0$ for some i. If $\kappa_i \neq 0$ for all i, then $\{\Psi_s(x_2), \Psi_s(x_3), \Psi_s(x_4)\}$ is a SAGBI basis for any monomial order $\preceq$ with $d_i \prec d_1$ for $i = 2, 3, 4$, where $s = x_1/D(x_1)$.*

Proof. The former part follows from Theorem 1.2. Assume that $D(x_i) \neq 0$ for any i. The assumption that $d_1 \prec d_i$ for $i = 2, 3, 4$ implies that $x_i = \mathrm{in}_{\preceq}(\Psi_s(x_i))$ for $i = 2, 3, 4$. Actually, $\mathrm{supp}(x_i^{-1} \Psi_s(x_i))$ is contained in $\sum_{i=1}^{3} \mathbf{R}_{\geq 0}(d_i - d_1)$ for each i. Hence, $\{\Psi_s(x_2), \Psi_s(x_3), \Psi_s(x_4)\}$ is a SAGBI basis for $\preceq$ by Corollary 3.4. $\square$

Assume that $m = 2$, and consider the k-derivation on $k[\mathbf{x}][\mathbf{y}]$ of the form

$$D = \mathbf{x}^{\delta_0}\frac{\partial}{\partial y_0} + \mathbf{x}^{\delta_1}\frac{\partial}{\partial y_1} + \mathbf{x}^{\delta_2}\frac{\partial}{\partial y_2}. \tag{3.31}$$

For each i, j, we set $\epsilon_{i,j}^{+}$ the vector $\epsilon_{i,j}$ whose negative components are replaced by zero, and define $L_{i,j} = \mathbf{x}^{\epsilon_{j,i}^{+}}y_i - \mathbf{x}^{\epsilon_{i,j}^{+}}y_j$. Khoury showed in [3, Corollary 2.2] that $k[\mathbf{x}][\mathbf{y}]^D$ is generated by $L_{2,1}, L_{3,1}, L_{3,2}$ over $k[\mathbf{x}]$. As a consequence of Theorem 3.2, we have furthermore the following result.

Corollary 3.10 *Assume that $m = 2$ and D is a k-derivation on $k[\mathbf{x}][\mathbf{y}]$ of the form* (3.31). *Then, $\{x_1, \ldots, x_n, L_{2,1}, L_{3,1}, L_{3,2}\}$ is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$.*

Proof. Note that (3.31) is a special case of (3.4) where $\kappa_0 = \kappa_1 = \kappa_{2,0} = 1$, $u_1 = u_2 = 1$, and $v = 0$. In this case, we have $\tilde{F} = y_1 - \mathbf{x}^{\epsilon_{1,0}}y_0$, $\tilde{G} = y_2 - \mathbf{x}^{\epsilon_{2,0}}y_0$, and $\eta = \delta_2 - \delta_1 = \epsilon_{2,1}$. Since $\epsilon_{2,1} + \epsilon_{1,0} = \epsilon_{2,0}$, we have

$$\tilde{H} = \mathbf{x}^{\epsilon_{2,1}}(y_1 - \mathbf{x}^{\epsilon_{1,0}}y_0) - (y_2 - \mathbf{x}^{\epsilon_{2,0}}y_0) = \mathbf{x}^{\epsilon_{2,1}}y_1 - y_2. \tag{3.32}$$

For i, j, it follows that $\rho(y_i - \mathbf{x}^{\epsilon_{i,j}}y_j) = \mathbf{x}^{\epsilon_{j,i}^{+}}y_i - \mathbf{x}^{\epsilon_{i,j}^{+}}y_j$. Therefore, the assertion follows from Theorem 3.2. $\square$

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Chapter 3

A generalization of Roberts' counterexample to the fourteenth problem of Hilbert

A generalization of Roberts' counterexample to the fourteenth problem of Hilbert

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Abstract

We generalize Roberts' counterexample to the fourteenth problem of Hilbert, and give a sufficient condition for certain invariant rings not to be finitely generated. It shows that there exist a lot of counterexamples of this type. We also determine the initial algebra of Roberts' counterexample for some monomial order.

1 Introduction

The fourteenth problem of Hilbert asks whether the K-algebra $L \cap A$ is finitely generated. Here, K is a field, A is a polynomial ring over K, and L is a subfield of the quotient field of A containing K. The first counterexample to this problem was found by Nagata in 1958. It was given as the invariant subring of a polynomial ring in 32 variables for a linear action of the 13-dimensional additive group [13]. Recently, Mukai showed that there exists similar counterexample which is an invariant subring of a polynomial ring in 18 variables for a linear action of the three dimensional additive group [12].

In 1990, P. Roberts gave a simple new counterexample of different type as follows.

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Theorem 1.1 (P. Roberts [15, Theorem 1]) *Let $A = K[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$ be a polynomial ring in seven variables over a field K of characteristic zero. Assume that L_t is a subfield of the quotient field of A generated by*

$$x_1, \quad x_2, \quad x_3, \quad x_1y_4 - x_2^tx_3^ty_1, \quad x_2y_4 - x_1^tx_3^ty_2, \quad x_3y_4 - x_1^tx_2^ty_3 \tag{1.1}$$

over K. If $t \geq 2$, then $L_t \cap A$ is not finitely generated over K.

Following this result, Deveney and Finston showed in [2] that this counterexample can be obtained as the invariant subring of A for a nonlinear action of the one dimensional additive group G_a. In [6], Kojima and Miyanishi generalized Roberts' counterexample. They constructed a G_a-invariant subring of the polynomial ring of each dimension greater than or equal to seven which is not finitely generated. Furthermore, Freudenburg gave a counterexample in dimension six [4], and Daigle and Freudenburg gave one in dimension five [1].

In the present paper, we will generalize Roberts' counterexample further, and show that there exist a lot of counterexamples of this type. We give in Theorems 1.3 and 1.4 sufficient conditions for a certain kind of G_a-invariant subring of a polynomial ring not to be finitely generated. In Section 3, we will discuss Roberts' counterexample $L_t \cap A$ in terms of the theory of SAGBI bases. As the consequence, we determine a generating set of it in Theorem 3.3. We also remark on a sufficient condition for finite generation in Section 4.

Throughout this paper, let K denote a field of characteristic zero. Assume that R is a commutative K-algebra, and A is a commutative R-algebra. An R-homomorphism $D : A \to A$ is called an *R-derivation* on A if $D(ab) = D(a)b + aD(b)$ holds for any $a, b \in A$. Then, its kernel

$$A^D = \{a \in A \mid D(a) = 0\} \tag{1.2}$$

is an R-subalgebra of A. An R-derivation D on A is said to be *locally nilpotent* if, for each $a \in A$, there exists $r \in \mathbf{Z}_{\geq 0}$ such that $D^r(a) = 0$. We remark that a locally nilpotent R-derivation D on A defines an action $A \to A \otimes_R R[t]$ of the one dimensional additive group scheme $G_a = \operatorname{Spec} R[t]$ over R on A by $a \mapsto \sum_{k\geq 0} D^k(a) \otimes (t^k/k!)$. The invariant subring A^{G_a} of A for this action of G_a is equal to A^D (cf. [11]).

Let $R = K[\mathbf{x}] = K[x_1, \ldots, x_m]$ be the polynomial ring in m variables over K, and $A = K[\mathbf{x}][\mathbf{y}] = K[\mathbf{x}][y_1, \ldots, y_n]$ that in n variables over $K[\mathbf{x}]$. A $K[\mathbf{x}]$-derivation D on $K[\mathbf{x}][\mathbf{y}]$ is said to be *elementary* if $D(y_j)$ is in $K[\mathbf{x}]$ for each j. Note that an elementary $K[\mathbf{x}]$-derivation is locally nilpotent. An elementary $K[\mathbf{x}]$-derivation D on $K[\mathbf{x}][\mathbf{y}]$ is said to be *monomial* if each $D(y_i)$ is a monomial, i.e., $x_1^{a_1} \cdots x_m^{a_n}$ for some $(a_1, \ldots, a_m) \in \mathbf{Z}_{\geq 0}{}^m$. In this paper, we discuss the problem of finite generation of the kernel $K[\mathbf{x}][\mathbf{y}]^D$ of such a $K[\mathbf{x}]$-derivation D. As we remarked above, it is equal to the invariant subring of $K[\mathbf{x}][\mathbf{y}]$ for an action of G_a, since D is locally nilpotent. Note that $K[\mathbf{x}][\mathbf{y}]^D$ is finitely generated over K if and only if it is so over $K[\mathbf{x}]$.

In case of $n = m + 1$, the $K[\mathbf{x}]$-derivation

$$D_{t,m} = x_1^{t+1}\frac{\partial}{\partial y_1} + \cdots + x_m^{t+1}\frac{\partial}{\partial y_m} + (x_1 \cdots x_m)^t \frac{\partial}{\partial y_{m+1}} \tag{1.3}$$

on $K[\mathbf{x}][\mathbf{y}]$ is elementary and monomial. The kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of this $K[\mathbf{x}]$-derivation has been studied well. Deveney and Finston treated it in [2], and showed that Roberts' K-algebra $L_t \cap A$ in Theorem 1.1 is equal to the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ for $m = 3$ (See also Maubach's result found in [3, Section 9.6]). Furthermore, Kojima and Miyanishi showed the following.

Theorem 1.2 (Kojima-Miyanishi [6]) *Assume that $n = m + 1$. If $t \geq 2$ and $m \geq 3$, then the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of the $K[\mathbf{x}]$-derivation $D_{t,m}$ is not finitely generated over K.*

We will study the kernel $K[\mathbf{x}][\mathbf{y}]^D$ of an elementary monomial $K[\mathbf{x}]$-derivation D on $K[\mathbf{x}][\mathbf{y}]$ of more general form. Let $D(y_i) = \mathbf{x}^{\delta_i}$ for each $i = 1, \ldots, n$. Here, we denote by $\mathbf{x}^a$ the monomial $x_1^{a_1} \cdots x_m^{a_m}$ for $a = (a_1, \ldots, a_m) \in \mathbf{Z}^m$. Similarly, by $\mathbf{y}^b$ the monomial $y_1^{b_1} \cdots y_n^{b_n}$ for $b = (b_1, \ldots, b_n) \in \mathbf{Z}^n$. Put $\epsilon_{i,j} = \delta_i - \delta_j$ for i, j, and for $k = 1, \ldots, m$, let $\epsilon_{i,j}^k$ and δ_i^k be the k-th components of $\epsilon_{i,j}$ and δ_i, respectively.

In Sections 1 and 2, we study under the condition that $n \geq 4$, $m \geq n-1$ and $\epsilon_{i,j}^i > 0$ for any $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$. The derivation $D_{t,m}$ satisfies this condition.

Actually, we have $\epsilon^i_{i,j} = t+1$ if $j \neq m+1$, and $\epsilon^i_{i,j} = 1$ otherwise for it. We define

$$\eta = \frac{\epsilon^1_{1,n}}{\min\{\epsilon^1_{1,j} \mid j = 2, \dots, n-1\}}, \tag{1.4}$$

and

$$\eta_{k,i} = \eta \min\{\max\{\epsilon^i_{1,k}, \epsilon^i_{2,k}\}, 0\} \tag{1.5}$$

for $i = 2, \dots, n-1$ and $k = 3, \dots, n-1$. For each $k = 3, \dots, n-1$, we set $\mathcal{L}_{k,n-2}$ the system of linear inequalities

$$\begin{cases} u_1 + \cdots + u_{n-2} = 1 \\ u_1 \geq \eta,\ \ u_i \geq 0\ (i = 2, \dots, n-2) \\ \displaystyle\sum_{j=1}^{n-2} \min\{\epsilon^i_{n,1}, \epsilon^i_{n,j+1}\} u_j + \eta_{k,i} \geq 0\ (i = 2, \dots, n-1) \end{cases} \tag{1.6}$$

in the $n-2$ variables $u_1, \dots, u_{n-2}$.

Here is our main result.

Theorem 1.3 *Assume that $n \geq 4$, $m \geq n-1$ and $\epsilon^i_{i,j} > 0$ for any $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$. If the system $\mathcal{L}_{k,n-2}$ of linear inequalities has a solution in $\mathbf{R}^{n-2}$ for each $k = 3, \dots, n-1$, then $K[\mathbf{x}][\mathbf{y}]^D$ is not finitely generated over K.*

By this theorem, we get the following simple criterion for $n = 4$.

Theorem 1.4 *Assume that $m \geq 3$, $n = 4$ and $\epsilon^i_{i,j} > 0$ for any $1 \leq i \leq 3$, $1 \leq j \leq 4$ with $i \neq j$. If*

$$\frac{\epsilon^1_{1,4}}{\min\{\epsilon^1_{1,2}, \epsilon^1_{1,3}\}} + \frac{\epsilon^2_{2,4}}{\min\{\epsilon^2_{2,3}, \epsilon^2_{2,1}\}} + \frac{\epsilon^3_{3,4}}{\min\{\epsilon^3_{3,1}, \epsilon^3_{3,2}\}} \leq 1, \tag{1.7}$$

then $K[\mathbf{x}][\mathbf{y}]^D$ is not finitely generated over K.

The examples of Roberts are included in the case of this theorem for $m = 3$. In case $(m, n) = (3, 4)$, there exist 2450001 derivations on $K[\mathbf{x}][\mathbf{y}]$ which satisfy (1.7) and $\gcd\{\mathbf{x}^{\delta_1}, \mathbf{x}^{\delta_2}, \mathbf{x}^{\delta_3}, \mathbf{x}^{\delta_4}\} = 1$ even if we impose the restriction $\delta^k_i \leq 10$ for all i, k.

In the following corollary, the case where $m \geq 4$ and $t = 1$ is new, while the case $m \geq 3$ and $t \geq 2$ was proved in [6].

Corollary 1.5 *Assume that $n = m + 1$. If $m \geq 3$ and $t \geq 2$, or $m \geq 4$ and $t = 1$, then the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of the $K[\mathbf{x}]$-derivation $D_{t,m}$ is not finitely generated over K.*

We will prove Theorems 1.3, 1.4 and Corollary 1.5 in Section 2.

We remark that, if $t = 0$, then the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of $D_{t,m}$ is finitely generated for any m by Weitzenböck's theorem (cf. [13, Chapter IV]). Actually, it is isomorphic to a polynomial ring in $2m$ variables over K (See the remark after Lemma 4.2). If $m \leq 2$, then $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ is also isomorphic to a polynomial ring in $2m$ variables over K for any $t \geq 0$ by [5, Theorem 3.1]. For $(t, m) = (1, 3)$, Kurano showed in [7] that $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ is generated by nine elements over $K[\mathbf{x}]$.

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2 Construction of invariants

In this section, we prove Theorem 1.3, and show Theorem 1.4 and Corollary 1.5 as its consequences. Throughout this section, we assume that $n \geq 4$, $m \geq n-1$ and D satisfies that $\epsilon^i_{i,j} > 0$ for any $1 \leq i \leq n-1$, $1 \leq j \leq n$ with $i \neq j$. Note that D is uniquely extended to a $K[\mathbf{x}, \mathbf{x}^{-1}]$-derivation on $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$.

Theorem 1.3 follows from the two lemmas below.

Lemma 2.1 *If an element of $K[\mathbf{x}][\mathbf{y}]^D$ has a monomial of the form $\mathbf{x}^a y_n^l$ with $l > 0$, then at least one of the first $n - 1$ components of $a \in \mathbf{Z}_{\geq 0}{}^m$ is positive.*

Proof. Suppose that $f \in K[\mathbf{x}][\mathbf{y}]^D$ had $\mathbf{x}^a y_n^l$ with the first $n-1$ components of a are zero. Then, the monomial $\mathbf{x}^a \mathbf{x}^{\delta_n} y_n^{l-1}$ appears in $D(f)$. Since $D(f) = 0$, its coefficient in $D(f)$ is zero. Hence, $\mathbf{x}^a \mathbf{x}^{\delta_n} y_n^{l-1}$ is a monomial of $D(\mathbf{x}^{a'} \mathbf{y}^{b'})$ for some monomial $\mathbf{x}^{a'} \mathbf{y}^{b'} \neq \mathbf{x}^a y_n^l$ of f. Such $\mathbf{x}^{a'} \mathbf{y}^{b'}$ must be equal to $\mathbf{x}^a \mathbf{x}^{\epsilon_{n,i}} y_i y_n^{l-1}$ for some $i < n$. Since $\epsilon^i_{n,i} < 0$ for $i < n$, we have $\mathbf{x}^{a'} \mathbf{y}^{b'} \notin K[\mathbf{x}][\mathbf{y}]$. This contradicts that $f \in K[\mathbf{x}][\mathbf{y}]$. Thus, at least one of the first $n - 1$ components of $a \in \mathbf{Z}_{\geq 0}{}^m$ is positive. $\square$

The lemma below asserts the existence of an infinite system of invariants.

Lemma 2.2 *Under the assumption in Theorem* 1.3*, there exists a positive integer* α *such that a Laurent polynomial of the form*

$$x_1^\alpha y_n^l + (\textit{terms of lower degree in } y_n) \tag{2.1}$$

belongs to $K[\mathbf{x}, x_n^{-1}, \ldots, x_m^{-1}][\mathbf{y}]^D$ *for each* $l > 0$.

First, we show Theorem 1.3 by assuming these lemmas. Suppose that $K[\mathbf{x}][\mathbf{y}]^D$ was generated by finite elements $g_1, \ldots, g_p$. Then, by Lemma 2.1, there exists $r > 0$ such that each monomial of the form $x_1^\beta \mathbf{x}^b y_n^l$ of g_i with $l > 0$ and the first $n-1$ components of b are zero satisfies $l/\beta < r$ for every i. Since every element of $K[\mathbf{x}][\mathbf{y}]^D$ is written as a sum of products of $g_1, \ldots, g_p$, a monomial of an element of $K[\mathbf{x}][\mathbf{y}]^D$ is a product of those of $g_1, \ldots, g_p$. Hence, that of the form $x_1^\beta \mathbf{x}^b y_n^l$ as above also satisfies $l/\beta < r$. By Lemma 2.2, there exists $f \in K[\mathbf{x}, x_n^{-1}, \ldots, x_m^{-1}][\mathbf{y}]^D$ involving $x_1^\alpha y_n^l$ with $l/\alpha > r$. Since $\mathbf{x}^a f$ is in $K[\mathbf{x}][\mathbf{y}]^D$ for some $a \in \mathbf{Z}_{\geq 0}{}^m$ whose first $n-1$ components are zero, we are led to a contradiction. Thus, $K[\mathbf{x}][\mathbf{y}]^D$ is not finitely generated.

Let us denote $K[\mathbf{y}] = K[y_1, \ldots, y_n]$, and $K[\mathbf{y}]_l$ the K-vector subspace of homogeneous l-forms in $y_1, \ldots, y_n$. For each $f = \sum_{b \in \mathbf{Z}^n} \lambda_b \mathbf{y}^b \in K[\mathbf{y}]$, we define the *support* $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) = \{b \in \mathbf{Z}^n \mid \lambda_b \neq 0\}. \tag{2.2}$$

For each $a \in \mathbf{Z}^m$ and $l \in \mathbf{Z}_{\geq 0}$, we define the K-linear map $\tau_{\mathbf{x}^a}^l : K[\mathbf{y}]_l \to K[\mathbf{x}, \mathbf{x}^{-1}] \otimes_K K[\mathbf{y}]_l$ by $\tau_{\mathbf{x}^a}^l(\mathbf{y}^b) = \mathbf{x}^{a'} \mathbf{y}^b$. Here, $b = (b_1, \ldots, b_n)$ and $a' = a + \sum_{j=1}^n b_j \epsilon_{n,j}$. We define the elementary K-derivation E on $K[\mathbf{y}]$ by

$$E = \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial y_n}. \tag{2.3}$$

Then, it follows that $D(\tau_{\mathbf{x}^a}^l(f)) = \mathbf{x}^{\delta_n} \tau_{\mathbf{x}^a}^{l-1}(E(f))$ for each a, l and $f \in K[\mathbf{y}]_l$. We set

$$B = K[y_2 - y_1, y_3 - y_1, \ldots, y_n - y_1]. \tag{2.4}$$

Note that B is contained in $K[\mathbf{y}]^E$. Put $B_l = B \cap K[\mathbf{y}]_l$ for each $l \in \mathbf{Z}_{\geq 0}$. Then, $\tau^l_{\mathbf{x}^a}(f)$ is in $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]^D$ for each $a \in \mathbf{Z}^m$ and $f \in B_l$. Actually, $D(\tau^l_{\mathbf{x}^a}(f)) = \mathbf{x}^{\delta_n}\tau^{l-1}_{\mathbf{x}^a}(E(f)) = 0$, since $E(f) = 0$ for $f \in B_l$. We define the $\mathbf{R}$-linear maps $l_i : \mathbf{R}^n \to \mathbf{R}$ by

$$l_1((b_1, \ldots, b_n)) = \epsilon^1_{n,1} b_1 + \min\{\epsilon^1_{n,j} \mid j = 2, \ldots, n-1\} \sum_{j=2}^{n-1} b_j \tag{2.5}$$

and

$$l_i((b_1, \ldots, b_n)) = \sum_{j=1}^{n-1} \min\{\epsilon^i_{n,1}, \epsilon^i_{n,j}\} b_j \tag{2.6}$$

for $i = 2 \ldots, n-1$.

We reduce Lemma 2.2 to the following lemma.

Lemma 2.3 *Under the assumption in Theorem* 1.3*, there exists a positive integer* α *such that, for each positive integer* l*, we may find* $f \in B_l$ *such that* $(0, \ldots, 0, l) \in \operatorname{supp}(f)$ *and every* $b \in \operatorname{supp}(f)$ *satisfies* $l_1(b) + \alpha \geq 0$ *and* $l_i(b) \geq 0$ *for* $i = 2, \ldots, n-1$.

Lemma 2.2 is proved by this lemma as follows. As we mentioned above, $\tau^l_{x_1^\alpha}(f)$ is an element of $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]^D$, which has the form of (2.1). By definition, every monomial of $\tau^l_{x_1^\alpha}(f)$ is written as $x_1^\alpha \mathbf{x}^{a'} \mathbf{y}^b$, where $b = (b_1, \ldots, b_n) \in \operatorname{supp}(f)$ and $a' = \sum_{j=1}^n b_j \epsilon_{n,j}$. By assumption, we have

$$\sum_{j=1}^n b_j \epsilon^1_{n,j} + \alpha \geq l_1(b) + \alpha \geq 0 \tag{2.7}$$

and

$$\sum_{j=1}^n b_j \epsilon^i_{n,j} \geq l_i(b) \geq 0 \tag{2.8}$$

for $i = 2, \ldots, n-1$. Hence, $x_1^\alpha \mathbf{x}^{a'} \mathbf{y}^b$ does not have negative power in $x_1, \ldots, x_{n-1}$. Thus, $\tau^l_{x_1^\alpha}(f)$ is in $K[\mathbf{x}, x_n^{-1}, \ldots, x_m^{-1}][\mathbf{y}]^D$. This proves Lemma 2.2.

Let P_D be the set of $b = (b_1, \ldots, b_n) \in {\mathbf{R}_{\geq 0}}^n$ with $b_1 = b_n = 0, \sum_{j=2}^{n-1} b_j = 1$ and $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$. For each $b = (b_1, \ldots, b_{n-2}) \in \mathbf{R}^{n-2}$, we set $\iota(b) = (0, b_1, \ldots, b_{n-2}, 0)$. Note that, if $b \in {\mathbf{R}_{\geq 0}}^{n-2}$ is a solution of $\mathcal{L}_{k,n-2}$, then $l_i(\iota(b)) + \eta_{k,i} \geq 0$ for $i = 2, \ldots, n-1$. This condition is equivalent to the condition that $\iota(b), \iota(b) + \eta(\mathbf{e}_k - \mathbf{e}_2) \in P_D$, where

$\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the coordinate unit vectors of $\mathbf{R}^n$. Actually, if $\epsilon^i_{n,k} < \epsilon^i_{n,1}$, then

$$\begin{aligned}
\eta_{k,i} &= \eta \min\{\max\{\epsilon^i_{1,k}, \epsilon^i_{2,k}\}, 0\} && (2.9)\\
&= \eta \min\{\epsilon^i_{n,k} - \min\{\epsilon^i_{n,1}, \epsilon^i_{n,2}\}, 0\} && (2.10)\\
&= \eta \min\{\min\{\epsilon^i_{n,k}, \epsilon^i_{n,1}\} - \min\{\epsilon^i_{n,1}, \epsilon^i_{n,2}\}, 0\} && (2.11)\\
&= \min\{\eta l_i(\mathbf{e}_k - \mathbf{e}_2), 0\}. && (2.12)
\end{aligned}$$

If $\epsilon^i_{n,k} \geq \epsilon^i_{n,1}$, then $\epsilon^i_{1,k} \geq 0$. The equality $\eta_{k,i} = \min\{\eta l_i(\mathbf{e}_k - \mathbf{e}_2), 0\}$ also holds in this case, since the right hand sides of both (2.9) and (2.11) are zero.

For a convex subset $P \subset \mathbf{R}^n$, we denote by rP the subset $\{rb \mid b \in P\}$ of $\mathbf{R}^n$ for $r \in \mathbf{R}_{\geq 0}$.

Lemma 2.4 *Under the assumption in Theorem* 1.3, *there exists* $\alpha' > 0$ *such that, for any* $r > \alpha'$ *and* $u_3, \ldots, u_{n-1} \geq 0$ *with* $\sum_{k=3}^{n-1} u_k \leq \eta(r - \alpha')$, *there exist* $p_3, \ldots, p_{n-1} \in \mathbf{Z}_{\geq 0}$ *such that*

$$r\mathbf{e}_2 + \sum_{k=3}^{n-1} (s_k u_k + p_k)(\mathbf{e}_k - \mathbf{e}_2) \in rP_D \tag{2.13}$$

for any $s_j \in [0, 1]$.

Proof. Since $\mathcal{L}_{k,n-2}$ has a solution, there exists $\mathbf{b}_k \in P_D$ such that $\mathbf{b}_k + \eta(\mathbf{e}_k - \mathbf{e}_{n-2})$ is in P_D for each $k = 3, \ldots, n-1$. Let P be the convex hull of

$$\{\mathbf{b}_k, \mathbf{b}_k + \eta(\mathbf{e}_k - \mathbf{e}_2) \mid k = 3, \ldots, n-1\} \tag{2.14}$$

in $\mathbf{R}^n$, and d a positive number such that the d-neighbourhood of a point $\mathbf{a} \in P$ is contained in P. Here, we consider the Euclidean topology induced from that on the affine subspace $H = \mathbf{e}_2 + \sum_{k=3}^{n-1} \mathbf{R}(\mathbf{e}_k - \mathbf{e}_2)$. Then, define $\alpha' = (1/d)\sqrt{(n-2)(n-3)}$. We show that this α' satisfies the desired property.

Take any $r > \alpha'$. It suffices to show (2.13) for $u_3, \ldots, u_{n-1} \geq 0$ with $\sum_{k=3}^{n-1} u_k = \eta(r - \alpha')$. We set $u'_k = u_k/(\eta(r - \alpha'))$ for each k. Then, we have

$$\sum_{k=3}^{n-1} u'_k(\mathbf{b}_k + s_k\eta(\mathbf{e}_k - \mathbf{e}_2)) \in P \tag{2.15}$$

for any $s_3, \ldots, s_{n-1} \in [0,1]$. Actually, since P is convex,

$$\mathbf{b}_k + s_k\eta(\mathbf{e}_k - \mathbf{e}_2) = (1-s_k)\mathbf{b}_k + s_k(\mathbf{b}_k + \eta(\mathbf{e}_k - \mathbf{e}_2)) \tag{2.16}$$

is in P. Since $\sum_{k=3}^{n-1} u'_k = 1$, we get (2.15).

For each $\mathbf{q} \in H$, define the map $T_{\mathbf{q}} : P \to rH$ by $T_{\mathbf{q}}(c) = \alpha'\mathbf{q} + (r-\alpha')c$. Since $0 < \alpha' < r$, we have $T_{\mathbf{q}}(P) \subset rP$ if $\mathbf{q} \in P$. Put $\mathbf{b}' = T_{\mathbf{a}}\left(\sum_{k=3}^{n-1} u'_k\mathbf{b}_k\right)$, and choose $p'_k \in \mathbf{R}_{\geq 0}$ so that $\mathbf{b}' = r\mathbf{e}_2 + \sum_{k=3}^{n-1} p'_k(\mathbf{e}_k - \mathbf{e}_2)$. Then, let p_k be the nonnegative integer obtained by adding an element in $(-1/2, 1/2]$ to p'_k for each k. Put $\mathbf{b} = r\mathbf{e}_2 + \sum_{k=3}^{n-1} p_k(\mathbf{e}_k - \mathbf{e}_2)$ and $\mathbf{a}' = \mathbf{a} + (\alpha')^{-1}(\mathbf{b} - \mathbf{b}')$. Then,

$$|\mathbf{b} - \mathbf{b}'| = \sqrt{\left(\sum_{k=3}^{n-1}(p_k - p'_k)\right)^2 + \sum_{k=3}^{n-1}(p_k - p'_k)^2} \leq \frac{\sqrt{(n-2)(n-3)}}{2}. \tag{2.17}$$

So, we have

$$|\mathbf{a} - \mathbf{a}'| = (\alpha')^{-1}|\mathbf{b} - \mathbf{b}'| \leq d/2. \tag{2.18}$$

By the choice of $\mathbf{a}$, the point $\mathbf{a}'$ is in P. Hence, $T_{\mathbf{a}'}(P) \subset rP$. Moreover, $T_{\mathbf{a}'}(c) - T_{\mathbf{a}}(c) = \alpha'(\mathbf{a}' - \mathbf{a}) = \mathbf{b} - \mathbf{b}'$ for $c \in P$. Thus, we get

$$(\mathbf{b} - \mathbf{b}') + T_{\mathbf{a}}(P) \subset rP. \tag{2.19}$$

By (2.15),

$$\mathbf{b} - \mathbf{b}' + T_{\mathbf{a}}\left(\sum_{k=3}^{n-1} u'_k(\mathbf{b}_k + s_k\eta(\mathbf{e}_k - \mathbf{e}_2))\right) = \mathbf{b} + \sum_{k=3}^{n-1} s_k u_k(\mathbf{e}_k - \mathbf{e}_2) \tag{2.20}$$

$$= r\mathbf{e}_2 + \sum_{k=3}^{n-1}(p_k + s_k u_k)(\mathbf{e}_k - \mathbf{e}_2) \tag{2.21}$$

is in $\mathbf{b} - \mathbf{b}' + T_{\mathbf{a}}(P)$ for any $s_k \in [0,1]$. Then, (2.13) follows from (2.19), since rP is contained in rP_D. Therefore, α' satisfies the desired property. □

Now, let us prove Lemma 2.3. First, we show that the assumption that each $\mathcal{L}_{k,n-2}$ has a solution implies that $\epsilon^i_{n,1} \geq 0$ and $\epsilon^1_{n,i} > 0$ for $i = 2, \ldots, n-1$. Suppose that $\epsilon^i_{n,1} < 0$

for some $2 \leq i \leq n-1$. Then, for any $(u_1, \ldots, u_{n-2}) \in \mathbf{R}_{\geq 0}{}^{n-2}$ with $\sum_{j=1}^{n-2} u_j = 1$, we have

$$\sum_{j=1}^{n-2} \min\{\epsilon_{n,1}^i, \epsilon_{n,j+1}^i\} u_j + \eta_{k,i} \leq \epsilon_{n,1}^i + \eta_{k,i} < 0. \tag{2.22}$$

This contradicts that $\mathcal{L}_{k,n-2}$ has a solution. Thus, $\epsilon_{n,1}^i \geq 0$ for $i = 2, \ldots, n-1$. Suppose that $\epsilon_{n,i}^1 \leq 0$ for some $2 \leq i \leq n-1$. Then, it implies that $\eta \geq 1$, since

$$\epsilon_{1,n}^1 - \min\{\epsilon_{1,j}^1 \mid j = 2, \ldots, n-1\} = -\min\{\epsilon_{n,j}^1 \mid j = 2, \ldots, n-1\} \geq -\epsilon_{n,i}^1 \geq 0. \tag{2.23}$$

If $\mathcal{L}_{k,n-2}$ has a solution $u = (u_1, \ldots, u_{n-2})$, then $\eta = u_1 = 1$ and $u_j = 0$ for $j = 2, \ldots, n-2$. For this u, it follows that

$$\sum_{j=1}^{n-2} \min\{\epsilon_{n,1}^2, \epsilon_{n,j+1}^2\} u_j + \eta_{k,2} = \min\{\epsilon_{n,1}^2, \epsilon_{n,2}^2\} + \eta_{k,2} \leq \epsilon_{n,2}^2 < 0. \tag{2.24}$$

This is a contradiction. Thus, $\epsilon_{n,i}^1 > 0$ for $i = 2, \ldots, n-1$.

Take $\alpha' > 0$ as in Lemma 2.4, and set α an integer greater than or equal to $\alpha' \epsilon_{1,n}^1$. Let l be an arbitrary positive integer, and $\mathcal{F}$ the set of $f \in B_l$ such that $(0, \ldots, 0, l) \in \operatorname{supp}(f)$ and every $b \in \operatorname{supp}(f)$ satisfies $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$. Since

$$l_i(j\mathbf{e}_1 + (l-j)\mathbf{e}_n) = j\epsilon_{n,1}^i \geq 0 \tag{2.25}$$

for $i = 2, \ldots, n-1$ and $j = 0, \ldots, l$, we have $(y_n - y_1)^l \in \mathcal{F}$. Hence, $\mathcal{F} \neq \emptyset$. We show that there exists $F_0 \in \mathcal{F}$ such that $l_1(b) + \alpha \geq 0$ for each $b \in \operatorname{supp}(F_0)$. Suppose the contrary. Then, for each $f \in \mathcal{F}$, an element $O(f) = (d, e)$ in $\mathbf{Z}^2$ is defined by setting d the maximum among the n-th components of $b \in \operatorname{supp}(f)$ with $l_1(b) + \alpha < 0$, and e the maximum among the first components of $b \in \operatorname{supp}(f)$ whose n-th components are d. We define the total order $\preceq$ on $\mathbf{Z}^2$ by $(d_1, e_1) \preceq (d_2, e_2)$ if $d_1 < d_2$ or $d_1 = d_2, e_1 \leq e_2$. For $v_1, v_2 \in \mathbf{Z}^2$, we denote by $v_1 \prec v_2$ if $v_1 \preceq v_2$ and $v_1 \neq v_2$. Choose $F \in \mathcal{F}$ with $O(F) = (d, e)$ such that $(d, e) \preceq O(h)$ for any $h \in \mathcal{F}$, and set $f \in K[y_2, \ldots, y_{n-1}]$ the coefficient of $y_1^e y_n^d$ in F.

For $b \in \mathrm{supp}(F)$ whose first and n-th components are e and d, respectively, we have

$$l_1(b) + \alpha = \epsilon^1_{n,1} e + \min\{\epsilon^1_{n,j} \mid j = 2, \ldots, n-1\}(l - d - e) + \alpha \tag{2.26}$$

$$= \epsilon^1_{n,1} e + (\epsilon^1_{n,1} + \min\{\epsilon^1_{1,j} \mid j = 2, \ldots, n-1\})(l - d - e) + \alpha \tag{2.27}$$

$$= \min\{\epsilon^1_{1,j} \mid j = 2, \ldots, n-1\}(l - d - e) - \epsilon^1_{1,n}(l - d) + \alpha \tag{2.28}$$

$$\geq \min\{\epsilon^1_{1,j} \mid j = 2, \ldots, n-1\}(l - d - e) - \epsilon^1_{1,n}(l - d - \alpha') \tag{2.29}$$

$$= \min\{\epsilon^1_{1,j} \mid j = 2, \ldots, n-1\}((l - d - e) - \eta(l - d - \alpha')). \tag{2.30}$$

Since $\epsilon^1_{1,j} > 0$ for $j \neq 1$, (2.28) is negative by the maximality of e. By (2.30) we get

$$l - d - e < \eta(l - d - \alpha'). \tag{2.31}$$

Lemma 2.5 *In the above notation, $E(f) = 0$.*

Proof. Suppose that $E(f) \neq 0$. Let $\mathbf{y}^b$ be a monomial appearing in $E(f)$ with nonzero coefficient. We set λ'_j the coefficient of $y_j \mathbf{y}^b$ in f, and b_j the j-th component of b for each j. Then, the coefficient μ' of $\mathbf{y}^b$ in $E(f)$ is written as

$$\mu' = \sum_{j=2}^{n-1} (b_j + 1)\lambda'_j. \tag{2.32}$$

We set λ_j the coefficient of $y_j \mathbf{y}^b (y_1^e y_n^d)$ in F for each j. Then, $\lambda_j = \lambda'_j$ for $j = 2, \ldots, n-1$. The coefficient μ of $\mathbf{y}^b (y_1^e y_n^d)$ in $E(F)$ is written as

$$\mu = (e+1)\lambda_1 + \sum_{j=2}^{n-1} (b_j + 1)\lambda_j + (d+1)\lambda_n = (e+1)\lambda_1 + \mu' + (d+1)\lambda_n. \tag{2.33}$$

Since $E(F) = 0$, we have $\mu = 0$. Moreover, $\lambda_1 = 0$ by the maximality of e. Since $\mu' \neq 0$, we have $\lambda_n \neq 0$, that is,

$$b' = b + e\mathbf{e}_1 + (d+1)\mathbf{e}_n \tag{2.34}$$

is in $\mathrm{supp}(F)$. Note that $l_1(b' + \mathbf{e}_2 - \mathbf{e}_n) + \alpha$ is negative, since it is equal to (2.26). Hence,

$$l_1(b') + \alpha = l_1(b' + \mathbf{e}_2 - \mathbf{e}_n) + l_1(\mathbf{e}_n - \mathbf{e}_2) \tag{2.35}$$

$$< l_i(\mathbf{e}_n - \mathbf{e}_2) = -\min\{\epsilon^1_{n,j} \mid j = 2, \ldots, n-1\} < 0. \tag{2.36}$$

This contradicts the maximality of d. Thus, we get $E(f) = 0$. $\square$

We claim that $K[\mathbf{y}]^E \subset B$. This is a special case of Lemma 4.2 which we shall prove later. By Lemma 2.5, this fact implies that f is in B_{l-d-e}.

Lemma 2.6 *In the above notation, there exists $G \in B_l$ of the form $G = fy_1^e y_n^d + g$. Here, g is an element of $K[\mathbf{y}]_l$ such that $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$ and, respectively, the first and n-th components e' and d' of b satisfy $(e', d') \prec (d, e)$ for every $b \in \operatorname{supp}(g)$.*

Proof. Since f is in $B_{l-d-e} \cap K[y_2, \ldots, y_{n-1}]$, we have

$$f = \sum_u \lambda_u \prod_{k=3}^{n-1} (y_2 - y_k)^{u_k} \tag{2.37}$$

for some $\lambda_u \in K$. Here, the sum in (2.37) is taken over $u = (u_3, \ldots, u_{n-1}) \in {\mathbf{Z}_{\geq 0}}^{n-3}$ with $\sum_{k=3}^{n-1} u_k = l - d - e$. By (2.31), we get $\sum_{k=3}^{n-1} u_k < \eta(l - d - \alpha')$ for each u. Hence, there exist $p_3, \ldots, p_{n-1} \in \mathbf{Z}_{\geq 0}$ such that

$$(l-d)\mathbf{e}_2 + \sum_{k=3}^{n-1} (s_k u_k + p_k)(\mathbf{e}_k - \mathbf{e}_2) \in (l-d)P_D \tag{2.38}$$

for any $0 \leq s_3, \ldots, s_{n-1} \leq 1$ by Lemma 2.4. We set

$$h'_u = y_2^{e-p} \prod_{k=3}^{n-1} \left((y_2 - y_k)^{u_k} y_k^{p_k} \right), \tag{2.39}$$

where $p = \sum_{k=3}^{n-1} p_k$. Note that each element of $\operatorname{supp}(h'_u)$ is written as the left of (2.38) for some $0 \leq s_k \leq 1$. So, $\operatorname{supp}(h'_u)$ is contained in $(l-d)P_D$. In particular, $e - p \geq 0$. We set

$$h_u = (y_1 - y_2)^{e-p} \prod_{k=3}^{n-1} \left((y_2 - y_k)^{u_k} (y_1 - y_k)^{p_k} \right) \tag{2.40}$$

for each u, and define

$$G = \left(\sum_u \lambda_u h_u \right) (y_n - y_1)^d. \tag{2.41}$$

Then, it is an element of B_l of the form $G = fy_1^e y_n^d + g$, where g is an element of $K[\mathbf{y}]_l$ such that, respectively, the first and n-th components e' and d' of each $b \in \operatorname{supp}(g)$ satisfy $(d', e') \prec (d, e)$. So, it suffices to verify $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$ for each $b \in \operatorname{supp}(G)$. Note that h_u is equal to h'_u which is substituted $y_1 - y_k$ for each y_k. Since

$\mathrm{supp}(h'_u) \subset (l-d)P_D$, each element of $\mathrm{supp}(h_u)$ is contained in $c + \sum_{j=2}^{n-1} \mathbf{Z}_{\geq 0}(\mathbf{e}_1 - \mathbf{e}_j)$ for some $c \in (l-d)P_D$. Therefore, we may write each $b \in \mathrm{supp}(G)$ as

$$b = d_1\mathbf{e}_1 + d_2\mathbf{e}_n + c + \sum_{j=2}^{n-1} v_j(\mathbf{e}_1 - \mathbf{e}_j), \tag{2.42}$$

where $d_1, d_2, v_2, \ldots, v_{n-1} \in \mathbf{Z}_{\geq 0}$ and $c \in (l-d)P_D$. Note that $l_i(\mathbf{e}_n) = 0$ and $l_i(\mathbf{e}_1), l_i(c) \geq 0$ for $i = 2, \ldots, n-1$. Moreover,

$$l_i\left(\sum_{j=2}^{n-1} v_j(\mathbf{e}_1 - \mathbf{e}_j)\right) = -\sum_{j=2}^{n-1} \min\{\epsilon^i_{n,1}, \epsilon^i_{n,j}\}v_j + \min\{\epsilon^i_{n,1}, \epsilon^i_{n,1}\}\sum_{j=2}^{n-1} v_j \tag{2.43}$$

$$= \sum_{j=2}^{n-1}(\epsilon^i_{n,1} - \min\{\epsilon^i_{n,1}, \epsilon^i_{n,j}\})v_j \geq 0 \tag{2.44}$$

for $i = 2, \ldots, n-1$. Thus, we get $l_i(b) \geq 0$ for $i = 2, \ldots, n-1$. This completes the proof. $\square$

We set $H = F - G$. Then, H is in $\mathcal{F}$. Moreover, $O(H) \prec O(F)$ by definition of H. This contradicts the choice of F. Hence, there exists $F_0 \in \mathcal{F}$ such that $l_1(b) + \alpha \geq 0$ for each $b \in \mathrm{supp}(F_0)$. We have thus proved Lemma 2.3. Therefore, the proof of Theorem 1.3 is completed.

Now, assume that $m \geq 3$ and $n = 4$. Then, we set

$$\xi_i = \xi_i(D) = \frac{\epsilon^i_{i,4}}{\min\{\epsilon^i_{i,j}, \epsilon^i_{i,k}\}} \tag{2.45}$$

for distinct integers $1 \leq i, j, k \leq 3$, and put $\xi(D) = \xi_1(D) + \xi_2(D) + \xi_3(D)$.

We show Theorem 1.4 as a consequence of Theorem 1.3. We verify that $(1 - \xi_2, \xi_2)$ is a solution of $\mathcal{L}_{3,2}$. Note that $\xi_i > 0$ for $i = 1, 2, 3$, $\eta = \xi_1$, $\eta_{3,2} = 0$ and $\eta_{3,3} = -\xi_1 \min\{\epsilon^3_{3,1}, \epsilon^3_{3,2}\}$. So, $\xi_2 > 0$. By (1.7), we have $1 - \xi_2 \geq \xi_1 + \xi_3 > \xi_1 = \eta$. Moreover, it follows that

$$\min\{\epsilon^2_{4,1}, \epsilon^2_{4,2}\}(1 - \xi_2) + \min\{\epsilon^2_{4,1}, \epsilon^2_{4,3}\}\xi_2 + \eta_{3,2} \tag{2.46}$$

$$= \min\{\epsilon^2_{4,1}, \epsilon^2_{4,2}\} + (\min\{\epsilon^2_{4,1}, \epsilon^2_{4,3}\} - \min\{\epsilon^2_{4,1}, \epsilon^2_{4,2}\})\xi_2 + \eta_{3,2} \tag{2.47}$$

$$= \epsilon^2_{4,2} + \min\{\epsilon^2_{2,1}, \epsilon^2_{2,3}\}\xi_2 = 0, \tag{2.48}$$

and

$$\min\{\epsilon_{4,1}^3, \epsilon_{4,2}^3\}(1-\xi_2) + \min\{\epsilon_{4,1}^3, \epsilon_{4,3}^3\}\xi_2 + \eta_{3,3} \tag{2.49}$$

$$= \min\{\epsilon_{4,1}^3, \epsilon_{4,2}^3\} + (\min\{\epsilon_{4,1}^3, \epsilon_{4,3}^3\} - \min\{\epsilon_{4,1}^3, \epsilon_{4,2}^3\})\xi_2 + \eta_{3,3} \tag{2.50}$$

$$= (\epsilon_{4,3}^3 + \min\{\epsilon_{3,1}^3, \epsilon_{3,2}^3\}) - \min\{\epsilon_{3,1}^3, \epsilon_{3,2}^3\}\xi_2 + \eta_{3,3} \tag{2.51}$$

$$= \min\{\epsilon_{3,1}^3, \epsilon_{3,2}^3\}\,(-\xi_3 + 1 - \xi_2 - \xi_1) \geq 0. \tag{2.52}$$

Therefore, $(1-\xi_2, \xi_2)$ is a solution of $\mathcal{L}_{3,2}$. Hence, $K[\mathbf{x}][\mathbf{y}]^D$ is not finitely generated by Theorem 1.3.

Finally, we show Corollary 1.5. As mentioned in Section 1, $\epsilon_{i,j}^i > 0$ for any $i \neq j$, since $\epsilon_{i,j}^i = t+1$ if $j \neq m+1$, and $\epsilon_{i,j}^i = 1$ otherwise. Assume that $m = 3$ and $t \geq 2$. Then, $\xi(D_{t,m}) = 3/(t+1) \leq 1$. Hence, $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$ is not finitely generated by Theorem 1.4.

Assume that $m \geq 4$ and $t \geq 1$. For $k = 3, \ldots, m-1$, we define $u_k = (u_k^1, \ldots, u_k^{m-1}) \in \mathbf{R}_{\geq 0}{}^{m-1}$ as follows. Set $u_3^3, u_k^j = 1/2$ for j, k with $j = 1$ or $k = j+2$, and set $u_k^j = 0$ otherwise. We show that u_k is a solution of $\mathcal{L}_{k,m-1}$ for each k. Since $m \geq 4$, we have $\sum_{j=1}^{m-1} u_k^j = 1$. Since $t \geq 1$, we get $u_k^1 = 1/2 \geq 1/(t+1) = \eta$. Clearly, $u_k^j \geq 0$ for $j = 2, \ldots, m-1$. For $i = 2, \ldots, m-1$, it follows that

$$\sum_{j=1}^{m-1} \min\{\epsilon_{m+1,1}^i, \epsilon_{m+1,j+1}^i\}u_k^j + \eta_{k,i} = t - (t+1)u_k^{i-1} + \eta_{k,i}. \tag{2.53}$$

Note that $\eta_{k,i} = -1$ if $i = k$, and $\eta_{k,i} = 0$ otherwise. If $i = k$, then the right hand side of (2.53) is equal to $t-1$, since $u_k^{k-1} = 0$. If $i \neq k$, then it is not less than $(t-1)/2$, since $u_k^{i-1} \leq 1/2$ for any i, k. So, it is nonnegative for every i, k. Therefore, u_k is a solution of $\mathcal{L}_{k,m-1}$ for $k = 3, \ldots, m-2$. By Theorem 1.3, $K[\mathbf{x}]^{D_{t,m}}$ is not finitely generated. Thus, we complete the proof of Corollary 1.5.

3 A SAGBI basis for the counterexample of Roberts

In this section, we consider the counterexample of Roberts. Recall that it is obtained as the kernel of the derivation $D_{t,m}$ on $K[\mathbf{x}][\mathbf{y}]$ for $(m, n) = (3, 4)$ and $t \geq 2$ by the result of Deveney and Finston [2]. We determine its initial algebra for some monomial order on $K[\mathbf{x}][\mathbf{y}]$. Consequently, it will turn out that an infinite system of invariants appeared in Roberts' proof of [15, Lemma 3] is a generating set of $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$.

First, we review the notion of an initial algebra and a SAGBI basis. Let $\preceq$ be a monomial order on $K[\mathbf{x}][\mathbf{y}]$, i.e., a total order on $\mathbf{Z}^m \times \mathbf{Z}^n$ such that $a \preceq b$ implies $a + c \preceq b + c$ for any $a, b, c \in \mathbf{Z}^m \times \mathbf{Z}^n$ and the zero vector is the minimum among ${\mathbf{Z}_{\geq 0}}^m \times {\mathbf{Z}_{\geq 0}}^n$ for $\preceq$. We denote by $a \prec b$ if $a \neq b$ and $a \preceq b$. We sometimes denote $\mathbf{x}^a\mathbf{y}^b \preceq \mathbf{x}^{a'}\mathbf{y}^{b'}$ instead of $(a, b) \preceq (a', b')$. For $f \in K[\mathbf{x}][\mathbf{y}] \setminus \{0\}$, we define the *initial term* $\mathrm{in}_\preceq(f)$ of f by $\alpha\mathbf{x}^a\mathbf{y}^b$. Here, (a, b) is the maximum element of $\mathrm{supp}(f)$ for $\preceq$, and α is the coefficient of $\mathbf{x}^a\mathbf{y}^b$ in f. Note that the maximum of $\mathrm{supp}(f)$ always exists, since it is a nonempty finite set. If $f = 0$, then we define $\mathrm{in}_\preceq(f) = 0$. Then, it follows that

$$\mathrm{in}_\preceq(fg) = \mathrm{in}_\preceq(f)\,\mathrm{in}_\preceq(g) \tag{3.1}$$

for any $f, g \in K[\mathbf{x}][\mathbf{y}]$. Assume that A is a K-subalgebra of $K[\mathbf{x}][\mathbf{y}]$. We define the *initial algebra* $\mathrm{in}_\preceq(A)$ of A by the K-vector space generated by $\{\mathrm{in}_\preceq(f) \mid f \in A\}$. Then, $\mathrm{in}_\preceq(A)$ is a K-algebra by (3.1). We say that a set $\mathcal{S} \subset A$ of generators of the K-algebra A is a *SAGBI basis* if the initial algebra $\mathrm{in}_\preceq(A)$ is generated by $\{\mathrm{in}_\preceq(f) \mid f \in \mathcal{S}\}$ over K.

The following is a basic property of a SAGBI basis.

Lemma 3.1 ([14, Proposition 1.16]) *Let $\preceq$ be a monomial order on $K[\mathbf{x}][\mathbf{y}]$. Assume that A is a K-subalgebra of $K[\mathbf{x}][\mathbf{y}]$, and $\mathcal{S}$ is a subset of A. If $\{\mathrm{in}_\preceq(f) \mid f \in \mathcal{S}\}$ generates the initial algebra $\mathrm{in}_\preceq(A)$ over K, then $\mathcal{S}$ is a SAGBI basis for A. In particular, $\mathcal{S}$ generates A over K.*

For any elementary monomial $K[\mathbf{x}]$-derivation D on $K[\mathbf{x}][\mathbf{y}]$, we set $\epsilon^+_{i,j}$ the same $\epsilon_{i,j}$ but the negative components are replaced by zero, and define $L_{i,j} = \mathbf{x}^{\epsilon^+_{j,i}}y_i - \mathbf{x}^{\epsilon^+_{i,j}}y_j$ for each i, j. Then, $L_{i,j}$ is in $K[\mathbf{x}][\mathbf{y}]^D$ for i, j.

Now, let us consider the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of $D_{t,m}$ on $K[\mathbf{x}][\mathbf{y}]$ for $(m, n) = (3, 4)$. Note that the following three elements

$$x_1^{t+1}y_2 - x_2^{t+1}y_1, \quad x_1^{t+1}y_3 - x_3^{t+1}y_1, \quad x_2^{t+1}y_3 - x_3^{t+1}y_2 \tag{3.2}$$

are contained in $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$. Actually, they are equal to $L_{2,1}, L_{3,1}$ and $L_{3,2}$. Moreover, we know the following (See also [6, Lemma 2.1]).

Theorem 3.2 (P. Roberts [15, Lemma 3]) *For each $d \in \mathbf{Z}_{\geq 0}$ and $i = 1, 2, 3$, there exists an element of the form $x_i y_4^d + (\text{terms of lower degree in } y_4)$ in $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$.*

We take an arbitrary $I_{d,i} \in K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$ of the form in Theorem 3.2 for each (d, i). Let $\preceq_{\text{lex}}$ be the monomial order on $K[\mathbf{x}][\mathbf{y}]$ for $(m, n) = (3, 4)$ which is the lexicographic order with

$$x_1 \prec_{\text{lex}} x_2 \prec_{\text{lex}} x_3 \prec_{\text{lex}} y_1 \prec_{\text{lex}} y_2 \prec_{\text{lex}} y_3 \prec_{\text{lex}} y_4. \tag{3.3}$$

Namely, we define $a \preceq_{\text{lex}} b$ if the last nonzero component of $b - a$ is positive for $a, b \in \mathbf{Z}^3 \times \mathbf{Z}^4$, where we regard a, b as elements of $\mathbf{Z}^7$.

The following is the main result of this section.

Theorem 3.3 *Assume that $t \geq 2$. Then, the initial algebra of $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$ for $\preceq_{\text{lex}}$ is generated by*

$$\{x_1^{t+1} y_2,\ x_1^{t+1} y_3,\ x_2^{t+1} y_3\} \cup \{x_i y_4^d \mid d \in \mathbf{Z}_{\geq 0},\ i = 1, 2, 3\} \tag{3.4}$$

over $K[\mathbf{x}]$. The union of

$$\{x_1, \quad x_2, \quad x_3, \quad x_1^{t+1} y_2 - x_2^{t+1} y_1, \quad x_1^{t+1} y_3 - x_3^{t+1} y_1, \quad x_2^{t+1} y_3 - x_3^{t+1} y_2\} \tag{3.5}$$

and $\{I_{d,i} \mid d \in \mathbf{Z}_{\geq 0}, i = 1, 2, 3\}$ is a SAGBI basis for $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$ for $\preceq_{\text{lex}}$. In particular, it generates $K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$ over K.

In case of $n = 3$, the set $\{x_1, \ldots, x_m, L_{2,1}, L_{3,1}, L_{3,2}\}$ is a SAGBI basis for the kernel $K[\mathbf{x}][\mathbf{y}]^D$ of an elementary monomial $K[\mathbf{x}]$-derivation D on $K[\mathbf{x}][\mathbf{y}]$ for any monomial order by [Chapter 2, Corollary 2.3]. We will use this fact in the proof of Theorem 3.3.

To analyze $K[\mathbf{x}][\mathbf{y}]^D$ in detail, we define a grading structure on it. Let D be any elementary monomial $K[\mathbf{x}]$-derivation on $K[\mathbf{x}][\mathbf{y}]$. We set

$$\Gamma = (\mathbf{Z}^m \times \mathbf{Z}^n) / \sum_{i=2}^{n} \mathbf{Z}(\epsilon_{i,1}, \mathbf{e}_1 - \mathbf{e}_i), \tag{3.6}$$

and $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$ the k-vector space generated by monomials $\mathbf{x}^a\mathbf{y}^b$ for $(a,b) \in \mathbf{Z}^m \times {\mathbf{Z}_{\geq 0}}^n$ with the image of (a,b) in Γ is equal to γ for each $\gamma \in \Gamma$. Then, it defines a Γ-grading on $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$, i.e., $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}] = \bigoplus_{\gamma\in\Gamma} K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$ and $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\mu \subset K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_{\gamma+\mu}$ for any $\gamma,\mu \in \Gamma$. Moreover, it follows that

$$K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]^D = \bigoplus_{\gamma\in\Gamma} K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]^D_\gamma. \tag{3.7}$$

Here, for a K-subalgebra A of $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$, we set $A_\gamma = A \cap K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for each γ. We say that $f \in K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$ is Γ-homogeneous if f is in $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for some $\gamma \in \Gamma$. This γ is denoted by $\deg_\Gamma(f)$. Note that each $\gamma \in \Gamma$ is expressed as a image of $(a, l\mathbf{e}_n)$ for some $a \in \mathbf{Z}^m$ and $l \in \mathbf{Z}_{\geq 0}$. Then, we have $\tau^l_{\mathbf{x}^a}(K[\mathbf{y}]_l) = K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$. Actually, $\tau^l_{\mathbf{x}^a}(\phi(f)) = f$ for $f \in K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]_\gamma$, where $\phi : K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}] \to K[\mathbf{y}]$ is the homomorphism which substitutes one for each x_i. In particular, we get $\tau^l_{\mathbf{x}^a}(B_l) = K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]^D_\gamma$.

We remark that, for $f \in K[\mathbf{y}]_l, r \in \mathbf{Z}_{\geq 0}$ and $a \in \mathbf{Z}^m$, the condition that $(y_i - y_j)^r$ divides f implies that $L^r_{i,j}$ is a factor of $\tau^l_{\mathbf{x}^a}(f)$ in $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$. Actually, the condition implies that $\tau^l_{\mathbf{x}^a}(f) = f' \prod_{k=1}^r (\mathbf{x}^{b_{i,k}} y_i - \mathbf{x}^{b_{j,k}} y_j)$ for some $f' \in K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$ and $b_{i,k}, b_{j,k} \in \mathbf{Z}^m$. Since $\tau^l_{\mathbf{x}^a}(f)$ is Γ-homogeneous, each $\mathbf{x}^{b_{i,k}} y_i - \mathbf{x}^{b_{j,k}} y_j$ must be equal to $\mathbf{x}^{c_k} L_{i,j}$ for some $c_k \in \mathbf{Z}^m$. So, $L^r_{i,j}$ is a factor of $\tau^l_{\mathbf{x}^a}(f)$ in $K[\mathbf{x},\mathbf{x}^{-1}][\mathbf{y}]$.

Assume that $n = 3$. Then, each $f \in B_l$ is written as

$$f = (y_2 - y_1)^s (y_3 - y_1)^t \sum_{i=0}^{u} \alpha_i (y_2 - y_1)^i (y_3 - y_1)^{u-i}. \tag{3.8}$$

Here, $s, t, u \in \mathbf{Z}_{\geq 0}$ with $s + t + u = l$ and $\alpha_i \in K$ with $\alpha_0, \alpha_u \neq 0$. If $\beta_1, \ldots, \beta_u \in \bar{K}$ are the solutions of the equation $\sum_{i=0}^u \alpha_i X = 0$, then we get

$$f = \alpha_0 (y_2 - y_1)^s (y_3 - y_1)^t \prod_{i=1}^{u} (y_2 - \beta_i y_3 + (\beta_i - 1) y_1), \tag{3.9}$$

where $\bar{K}$ is the algebraic closure of K. Therefore, each element of B_l is factored into the product of l elements in $\bar{K} \otimes_K B_1$. We note that, if r is the maximal integer such that $(y_3 - y_2)^r$ divides f, then the expansion of f involves the monomials $y_1^{l-r} y_2^r, y_1^{l-r} y_3^r$ and does not involve $y_1^{l-r'} y_2^{r'}, y_1^{l-r'} y_3^{r'}$ for $0 \leq r' \leq r$.

Lemma 3.4 *Assume that* $(m,n)=(3,3)$ *and* $\epsilon^i_{i,j}>0$ *for any* $1\le i,j\le 3$ *with* $i\ne j$*. If* $\gamma=\deg_\Gamma(L^p_{2,1}L^q_{3,1}L^r_{3,2})$ *for* $p,q,r\in\mathbf{Z}_{\ge 0}$*, then* $K[\mathbf{x}][\mathbf{y}]^D_\gamma$ *is equal to the one dimensional* K*-vector space generated by* $L^p_{2,1}L^q_{3,1}L^r_{3,2}$*.*

Proof. Take any $0\ne F\in K[\mathbf{x}][\mathbf{y}]^D_\gamma$. Then, there exists $f\in B_l$ such that $\tau^l_{\mathbf{x}^a}(f)=F$. Here, $l=p+q+r$ and $a=p(\epsilon_{2,3}+\epsilon^+_{1,2})+q\epsilon^+_{1,3}+r\epsilon^+_{2,3}$. If F was not in $K(L^p_{2,1}L^q_{3,1}L^r_{3,2})$, then $(y_2-y_1)^p$ or $(y_3-y_1)^q$ or $(y_3-y_2)^r$ would not divide f, as mentioned above. Suppose, say, that the maximal integer r' such that $(y_3-y_2)^{r'}$ divides f was less than r. Then, $y_1^{l-r'}y_2^{r'}$ and $y_1^{l-r'}y_3^{r'}$ appear in f with nonzero coefficient, as we mentioned above. Hence, so do $\tau^l_{\mathbf{x}^a}(y_1^{l-r'}y_2^{r'})$ and $\tau^l_{\mathbf{x}^a}(y_1^{l-r'}y_3^{r'})$ in F. By definition, the first component of $\epsilon^+_{2,3}$ or $\epsilon^+_{3,2}$ is zero. If that of $\epsilon^+_{2,3}$ is zero, then the power of x_1 in $\tau^l_{\mathbf{x}^a}(y_1^{l-r'}y_3^{r'})$ is negative. Actually, $\tau^l_{\mathbf{x}^a}(y_1^{l-r'}y_3^{r'})=\mathbf{x}^{a'}y_1^{l-r'}y_3^{r'}$, where

$$a'=a+(l-r')\epsilon_{3,1}=p\epsilon^+_{2,1}+q\epsilon^+_{3,1}+r\epsilon^+_{2,3}-(r-r')\epsilon_{1,3}. \tag{3.10}$$

Since the first components of $\epsilon^+_{2,1},\epsilon^+_{3,1},\epsilon^+_{2,3}$ are zero, that of a' is equal to $-(r-r')\epsilon^1_{1,3}<0$. Similarly, the power of x_1 in $\tau^l_{\mathbf{x}^a}(y_1^{l-r'}y_2^{r'})$ is negative if the first component of $\epsilon^+_{3,2}$ is zero. This is a contradiction. Therefore, F is in $K(L^p_{2,1}L^q_{3,1}L^r_{3,2})$. $\square$

Assume that $n=4$. We define the homomorphism $\tilde{l}:\mathbf{Z}^4\to\mathbf{Z}$ of additive groups by

$$\tilde{l}((b_1,b_2,b_3,b_4))=b_2\epsilon^1_{1,2}+b_3\epsilon^1_{1,3}. \tag{3.11}$$

Lemma 3.5 *Assume that* $n=4$*,* $\epsilon^1_{1,2}\ge\epsilon^1_{1,3}>0$ *and* f *is an element of* B_l *for some* $l\in\mathbf{Z}_{\ge 0}$*. If every* $b\in\operatorname{supp}(F)$ *satisfies that* $\tilde{l}(b)\ge p$ *for some* $p\in\mathbf{Z}_{\ge 0}$*, then* $(y_3-y_2)^q$ *divides* F *for the minimal* $q\in\mathbf{Z}_{\ge 0}$ *with* $p\le q\epsilon^1_{1,3}$*.*

Proof. Write

$$F=f_0(y_4-y_1)^l+f_1(y_4-y_1)^{l-1}+\cdots+f_l, \tag{3.12}$$

where $f_i\in K[y_2-y_1,y_3-y_1]_i$ for each i. Suppose that $(y_3-y_2)^q$ did not divide F. Then, there exists i such that $(y_3-y_2)^q$ does not divide f_i. Let i be the minimum among such indices i, and q' the maximal integer such that $(y_3-y_2)^{q'}$ divides f_i. Then, f_i involves the monomial $y_1^{i-q'}y_3^{q'}$, as we noted before Lemma 3.4. We set $b = (i - q', 0, q', l - i)$. Then, $f_j(y_4 - y_1)^{l-j}$ does not involve $\mathbf{y}^b$ if $j > i$, since the exponent of y_4 in each monomial of it is less than $l - i$. It also holds if $j < i$. Actually, if it did not, then f_j would contain $y_1^{j-q'}y_3^{q'}$. Since $q' < q$, this contradicts that $(y_3 - y_2)^q$ divides f_j by the note above. Therefore, $\mathbf{y}^b$ appears in F with nonzero coefficient. However, $\tilde{l}(b) = q'\epsilon_{1,3}^1 < q\epsilon_{1,3}^1$. It implies that $\tilde{l}(b) < p$ by the minimality of q. This contradicts that $b \in \mathrm{supp}(F)$. Therefore, $(y_3 - y_2)^q$ divides F. $\square$

We remark that, if $F \in K[\mathbf{x}][\mathbf{y}]^D$ is expressed as

$$F = f_0 y_n^l + f_1 y_n^{l-1} + \cdots + f_l \tag{3.13}$$

for $f_i \in K[\mathbf{x}][y_1, \ldots, y_{n-1}]$, then $D(f_0) = 0$. Actually, we get

$$0 = D(F) = D(f_0)y_n^l + (\text{terms of lower degree in } y_n). \tag{3.14}$$

The following is the key proposition.

Proposition 3.6 *Assume that* $(m, n) = (3, 4)$ *and* $\epsilon_{i,j}^i > 0$ *for any* $1 \le i \ne j \le 4$. *Then, the monomial* $\mathbf{x}^a y_2^p y_3^{q+r} y_4^l$ *is not contained in* $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\mathbf{x}][\mathbf{y}]^D)$ *for any* $p, q, r, l \in \mathbf{Z}_{\ge 0}$, *where we set* $a = p\epsilon_{1,2}^+ + q\epsilon_{1,3}^+ + r\epsilon_{2,3}^+$.

Proof. Suppose that there existed $F \in K[\mathbf{x}][\mathbf{y}]^D$ such that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F) = \mathbf{x}^a y_2^p y_3^{q+r} y_4^l$. Then, without loss of generality, we may assume that F is Γ-homogeneous. Write

$$F = f_0 y_4^l + f_1 y_4^{l-1} + \cdots + f_l, \tag{3.15}$$

where $f_i \in K[\mathbf{x}][y_1, y_2, y_3]$ for $i = 0, \ldots, l$. Then, f_0 is in $K[\mathbf{x}][y_1, y_2, y_3]^D$, as we remarked above. Moreover, f_0 is Γ-homogeneous and $\deg_\Gamma(f_0) = \deg_\Gamma(L_{1,2}^p L_{1,3}^q L_{2,3}^r)$. Hence, it is equal to $L_{1,2}^p L_{1,3}^q L_{2,3}^r$ up to a scalar multiplication by Lemma 3.4.

It suffices to show that each $L_{2,1}^p$, $L_{3,1}^q$ and $L_{3,2}^r$ must be a factor of F in $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$. Actually, it will imply that $F = L_{1,2}^p L_{1,3}^q L_{2,3}^r F'$ for some $F' \in K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$, since $L_{2,1}$, $L_{3,1}$ and $L_{3,2}$ are mutually prime. Then, F' is clearly an element in $K[\mathbf{x}][\mathbf{y}]^D$. However, it involves the monomial y_4^l. This contradicts Lemma 2.1.

Since the arguments are similar, we only show that $L^r_{3,2}$ is a factor of F. We assume that $\epsilon^1_{1,2} \geq \epsilon^1_{1,3}$. The proof is similar for the other case. We set $f = \phi(F)$. We claim that every $b = (b_1, \dots, b_4) \in \operatorname{supp}(f)$ satisfies $\tilde{l}(b) \geq r\epsilon^1_{1,3}$. By Lemma 3.5, it implies that $(y_3 - y_2)^r$ divides f. Thus, $L^r_{3,2}$ is a factor of F in $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$. By straightforward computation, we may verify that $\deg_\Gamma(F)$ is equal to the image of $(c, (d+l)\mathbf{e}_4)$, where $d = p + q + r$ and

$$c = p\epsilon^+_{2,1} + q\epsilon^+_{3,1} + r\epsilon^+_{2,3} + d\epsilon_{1,4} + r\epsilon_{3,1}. \tag{3.16}$$

So, it follows that $F = \tau^{d+l}_{\mathbf{x}^c}(f)$, as mentioned above. Hence, $\tau^{d+l}_{\mathbf{x}^c}(\mathbf{y}^b)$ is a monomial of F. By simple computation, we get $\tau^{d+l}_{\mathbf{x}^c}(\mathbf{y}^b) = \mathbf{x}^d\mathbf{y}^b$, where

$$d = p\epsilon^+_{2,1} + q\epsilon^+_{3,1} + r\epsilon^+_{2,3} + (l - b_4)\epsilon_{4,1} + r\epsilon_{3,1} + b_2\epsilon_{1,2} + b_3\epsilon_{1,3}. \tag{3.17}$$

Note that the first components of $\epsilon^+_{2,1}, q\epsilon^+_{3,1}, r\epsilon^+_{2,3}$ are zero and $b_4 \leq l$. Since $\mathbf{x}^d\mathbf{y}^b$ is in $K[\mathbf{x}][\mathbf{y}]$, the first component of d is nonnegative. Thus, we have

$$0 \leq (l - b_4)\epsilon^1_{4,1} + r\epsilon^1_{3,1} + b_2\epsilon^1_{1,2} + b_3\epsilon^1_{1,3} = (l - b_4)\epsilon^1_{4,1} - r\epsilon^1_{1,3} + \tilde{l}(b) < -r\epsilon^1_{1,3} + \tilde{l}(b). \tag{3.18}$$

Therefore, $\tilde{l}(b) \geq r\epsilon^1_{1,3}$. Thus, the proof is completed. □

Now, let us prove Theorem 3.3. By Lemma 3.1, the last statement is a consequence of the first part. So, we will prove the first part.

We set R the $K[\mathbf{x}]$-algebra generated by (3.4). Clearly, $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\mathbf{x}][\mathbf{y}]^{D_{t,3}})$ contains R. For the converse, it suffices to show that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F)$ is in R for any Γ-homogeneous element $F \in K[\mathbf{x}][\mathbf{y}]^{D_{t,3}}$. The remark before Proposition 3.6 implies that $\mathrm{in}_{\preceq_{\mathrm{lex}}}(F) = \mathrm{in}_{\preceq_{\mathrm{lex}}}(F')y_4^l$ for some $F' \in K[\mathbf{x}][y_1, y_2, y_3]^{D_{t,3}}$. By [Chapter 2, Corollary 2.3], the set $\{x_1, x_2, x_3, L_{2,1}, L_{3,1}, L_{3,2}\}$ is a SAGBI basis for $K[\mathbf{x}][y_1, y_2, y_3]^{D_{t,3}}$ with respect to any monomial order. In particular,

$$\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\mathbf{x}][y_1, y_2, y_3]^{D_{t,3}}) = K[\mathbf{x}][x_1^{t+1}y_2, x_1^{t+1}y_3, x_2^{t+1}y_3]. \tag{3.19}$$

Hence, there exist $a_1, a_2, a_3, p, q, r \in \mathbf{Z}_{\geq 0}$ such that

$$\mathrm{in}_{\preceq_{\mathrm{lex}}}(F) = (x_1^{t+1}y_2)^p(x_1^{t+1}y_3)^q(x_2^{t+1}y_3)^r x_1^{a_1}x_2^{a_2}x_3^{a_3}y_4^l. \tag{3.20}$$

If $l = 0$, then it is clearly in R. Assume that $l > 0$. Then, $a_1 + a_2 + a_3 > 0$ by Proposition 3.6. Hence, it is also in R. Therefore, $\mathrm{in}_{\preceq_{\mathrm{lex}}}(K[\mathbf{x}][\mathbf{y}]^{D_{t,3}})$ is contained in R. This completes the proof of Theorem 3.3.

4 A condition for finite generation

In this section, we investigate a condition for finite generation of $K[\mathbf{x}][\mathbf{y}]^D$, where D is an elementary monomial $K[\mathbf{x}]$-derivation. The main result of this section is the following.

Theorem 4.1 *Assume that $(m, n) = (3, 4)$ and, for every permutations σ and τ, respectively, on $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$, there exist $i \neq j$ and k such that $\epsilon^{\sigma(k)}_{\tau(i),\tau(j)} \leq 0$ and $\sigma(k) = \tau(i)$. Then, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by L_{k_i,l_i} for $i = 1, 2, 3, 4$ over $K[\mathbf{x}]$ for some integers $1 \leq k_i, l_i \leq 4$.*

First, we notice some general properties on the kernel of an elementary monomial $K[\mathbf{x}]$-derivation. For each i, j, we set $\tilde{L}_{j,i} = y_j - \mathbf{x}^{\epsilon_{j,i}} y_i$. If it is confusing, then we denote it by $\tilde{L}^D_{j,i}$ to emphasis D. Note that $\tilde{L}_{j,i}$ is contained in $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]^D$ for any i, j.

Lemma 4.2 *The kernel $K[\mathbf{x}][\mathbf{y}]^D$ is contained in $K[\mathbf{x}][\tilde{L}_{1,j}, \dots, \tilde{L}_{n,j}]$ for each j.*

Proof. Take any $F \in K[\mathbf{x}][\mathbf{y}]^D$, and set f the polynomial F which is substituted zero for y_j. Then, define an element F' of $K[\mathbf{x}][\tilde{L}_{1,j}, \dots, \tilde{L}_{n,j}]$ as f which is substituted $\tilde{L}_{k,j}$ for each y_k. We show that $F = F'$. Suppose that $F \neq F'$. Write

$$F - F' = (\text{terms of higher degree in } y_j) + g y_j^e, \tag{4.1}$$

where $g \in K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \setminus \{0\}$ not involving y_j. Then, $e > 0$. Actually, $F - f$ and $F' - f$ are in $K[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] y_j$. It follows that

$$0 = D(F - F') = (\text{terms of higher degree in } y_j) + eg\mathbf{x}^{\delta_j} y_j^{e-1}. \tag{4.2}$$

Since $eg\mathbf{x}^{\delta_j} \neq 0$, it is a contradiction. Therefore, $F = F'$. This completes the proof. □

Assume that $\delta_j = 0$ for some j. Then, $\tilde{L}_{k,j}$ is in $K[\mathbf{x}][\mathbf{y}]^D$ for any k. By Lemma 4.2, it implies that $K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][\tilde{L}_{1,j}, \ldots, \tilde{L}_{n,j}]$. If this is the case, then $K[\mathbf{x}][\mathbf{y}]^D$ is isomorphic to $K[\mathbf{x}][y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n]$ via the homomorphism which substitutes zero for y_j. In particular, the kernel $K[\mathbf{x}][\mathbf{y}]^{D_{t,m}}$ of the derivation $D_{t,m}$ for $t = 0$ is generated by $\tilde{L}_{1,m+1}, \ldots, \tilde{L}_{m,m+1}$ over $K[\mathbf{x}]$, and is isomorphic to the polynomial ring in $2m$ variables over K.

Now, we fix $1 \le i \le m$ and $1 \le j \le n$. Assume that $\epsilon^i_{k,j} \ge 0$ for every $k = 1, \ldots, n$. Put $\mu = \min\{\epsilon^i_{k,j} \mid k \ne j\}$, and set $\mathbf{x}^{\epsilon'_{k,j}} = x_i^{-\mu}\mathbf{x}^{\epsilon_{k,j}}$ for each k. Then, let D' be an elementary monomial $K[\mathbf{x}]$-derivation on $K[\mathbf{x}][\mathbf{y}]$ such that $D'(y_k)/D'(y_j) = \mathbf{x}^{\epsilon'_{k,j}}$ for each k. For $f \in K[\mathbf{x}][\mathbf{y}]^D$, we define $T_{j,i}(f)$ to be f which is substituted $x_i^{-\mu}y_j$ for y_j. Then, it follows that

$$T_{j,i}(\tilde{L}^D_{k,j}) = y_k - \mathbf{x}^{\epsilon_{k,j}}(x_i^{-\mu}y_j) = y_k - \mathbf{x}^{\epsilon'_{k,j}}y_j = \tilde{L}^{D'}_{k,j} \tag{4.3}$$

for each k.

Lemma 4.3 *Let i, j be integers with $1 \le i \le m$ and $1 \le j \le n$. We assume $\epsilon^i_{k,j} \ge 0$ for every $k = 1, \ldots, n$. Then, $T_{j,i}$ is an injective homomorphism with the image $K[\mathbf{x}][\mathbf{y}]^{D'}$.*

Proof. Suppose that $T_{j,i}(f)$ was not $K[\mathbf{x}][\mathbf{y}]^{D'}$ for some $f \in K[\mathbf{x}][\mathbf{y}]^D$. By Lemma 4.2, f is in $K[\mathbf{x}][\{\tilde{L}^D_{k,j} \mid k\}]$. Since $T_{j,i}$ sends $\tilde{L}^D_{k,j}$ to $\tilde{L}^{D'}_{k,j}$, we have $T_{j,i}(f) \in K[\mathbf{x}][\{\tilde{L}^{D'}_{k,j} \mid k\}]$. In particular, $D'(T_{j,i}(f)) = 0$. Hence, a monomial of $T_{j,i}(f)$ has a negative power in some variable. By definition of $T_{j,i}(f)$, the variable must be x_i. However, $\tilde{L}^{D'}_{k,j}$ does not have negative power in x_i for each k. Hence, such a monomial can not be involved in $T_{j,i}(f)$. This is a contradiction. Thus, $T_{j,i}(f)$ is in $K[\mathbf{x}][\mathbf{y}]^{D'}$.

Conversely, a homomorphism $K[\mathbf{x}][\mathbf{y}]^{D'} \to K[\mathbf{x}][\mathbf{y}]^D$ is defined by the substitution $y_j \mapsto x_i^{\mu}y_j$. Actually, it sends each $\tilde{L}^{D'}_{k,j}$ to $\tilde{L}^D_{k,j}$. It is the inverse of $T_{j,i} : K[\mathbf{x}][\mathbf{y}]^D \to K[\mathbf{x}][\mathbf{y}]^{D'}$. This proves the lemma. $\square$

The following proposition is used to reduce problems of the kernel of D to a lower dimensional case.

Proposition 4.4 *Let D be any elementary monomial $K[\mathbf{x}]$-derivation on $K[\mathbf{x}][\mathbf{y}]$, and $1 \leq j, k \leq m$ distinct integers. For each $1 \leq i \leq m$, we assume that either $\epsilon^i_{j,k} \geq 0$ or $\epsilon^i_{l,k} \geq 0$ for all $l \neq j$. Then, it follows that*

$$K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]^D[L_{j,k}]. \tag{4.4}$$

Proof. Clearly, the right hand side of (4.4) is contained in the left. We show the converse. Let S be the set of elements of $K[\mathbf{x}][\mathbf{y}]^D$ not contained in the left hand side of (4.4). Suppose that S was not empty. Take $f \in S$ with the minimal degree in y_j, and write

$$f = g_d(\mathbf{x}^{\epsilon^+_{k,j}} y_j)^d + g_{d-1}(\mathbf{x}^{\epsilon^+_{k,j}} y_j)^{d-1} + \cdots + g_0, \tag{4.5}$$

where $g_i \in K[\mathbf{x}, \mathbf{x}^{-1}][y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n]$ with $g_d \neq 0$. To complete the proof, it suffices to show that g_d is in $K[\mathbf{x}][\mathbf{y}]^D$. Actually, it implies that $f - g_d(L_{j,k})^d$ is in S, but the degree of $f - g_d(L_{j,k})^d$ in y_j is less than d. This is a contradiction, and we get $S = \emptyset$.

Similarly to the remark before Proposition 3.6, we have $D(g_d) = 0$. We show that every monomial of g_d does not have negative power in x_i for each i. If the i-th component of $\epsilon^+_{k,j}$ is zero, then it is true. Assume the contrary. Then, it is equal to $\epsilon^i_{k,j}$, and $\epsilon^i_{j,k}$ is negative. Hence, $\epsilon^i_{l,k} \geq 0$ for any $l \neq j$ by assumption. Since $\epsilon^i_{l,j} = \epsilon^i_{l,k} + \epsilon^i_{k,j}$, we have $0 < \epsilon^i_{k,j} \leq \epsilon^i_{l,j}$ for $l \neq j$. So, the substitution $y_j \mapsto x_i^{-\epsilon^i_{k,j}} y_j$ sends f to $T_{j,i}(f)$. If g_d had a monomial $\mathbf{x}^a \mathbf{y}^b$ with negative power in x_i, then $T_{j,i}(f)$ would have the monomial $\mathbf{x}^a \mathbf{y}^b y_j^d$. It also has negative power in x_i. This is a contradiction, since $T_{j,i}(f)$ is in $K[\mathbf{x}][\mathbf{y}]$ by Lemma 4.3. Therefore, no monomial of g_d has negative power in x_i for each i. Namely, g_d is in $K[\mathbf{x}][\mathbf{y}]$. This completes the proof. □

As a corollary to Proposition 4.4, we have the following.

Corollary 4.5 ([5, Theorem 3.1]) *Assume that $m = 2$. Then, there exist $1 \leq l \leq n$ and $1 \leq k_j \leq n$ with $k_j \neq j$ for each $j \neq l$ such that*

$$K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][L_{1,k_1}, \dots, L_{l-1,k_{l-1}}, L_{l+1,k_{l+1}}, \dots, L_{n,k_n}]. \tag{4.6}$$

Proof. We prove by induction on n. If $n = 1$, then $K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}]$ by Lemma 4.2. Hence, the assertion is true. Assume that $n > 1$. Then, by a change of indices, we may assume that $\delta_1^1 \leq \cdots \leq \delta_n^1$. If there exist $1 \leq k < j \leq n$ such that $\delta_k^2 \leq \delta_j^2$, then $\epsilon_{j,k}^i \geq 0$ for $i = 1, 2$. Hence,

$$K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n]^D[L_{j,k}] \tag{4.7}$$

by Proposition 4.4. Thus, the assertion follows from the induction assumption. Assume that such k, j do not exist, i.e., $\delta_n^2 < \cdots < \delta_1^2$. Then, $\epsilon_{l,n-1}^2 > 0$ for any $l \neq n$. Since $\epsilon_{n,n-1}^1 \geq 0$, we have $K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][y_1, \ldots, y_{n-1}]^D[L_{n,n-1}]$ by Proposition 4.4. Hence, the assertion follows similarly. Thus, the corollary is proved. $\square$

Let $\phi_1 : K[\mathbf{x}][\mathbf{y}] \to K[x_2, \ldots, x_m][\mathbf{y}]$ be the homomorphism which substitutes one for x_1, and D_1 the elementary $K[x_2, \ldots, x_m]$-derivation on $K[x_2, \ldots, x_m][\mathbf{y}]$ defined by $D_1(f) = \phi_1(D(f))$ for each f. Then, D_1 is monomial. By definition, it follows that $\phi_1 \circ D = D_1 \circ \phi_1$ on $K[\mathbf{x}][\mathbf{y}]$. Recall that a Γ-grading structure is defined on $K[\mathbf{x}][\mathbf{y}]$. We set Γ_1 the set of the image of $(a, l\mathbf{e}_n)$ in Γ for $l \in \mathbf{Z}$ and $a = (a_1, \ldots, a_m) \in \mathbf{Z}^m$ with $a_1 = 0$. It is a subgroup of Γ. Then, $\bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma$ is a $K[x_2, \ldots, x_n]$-subalgebra of $K[\mathbf{x}][\mathbf{y}]$.

Lemma 4.6 *Assume that $\epsilon_{n,j}^1 \geq 0$ for $j = 1, \ldots, n$. Then, ϕ_1 induces the isomorphism*

$$\bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma^D \to k[x_2, \ldots, x_n][\mathbf{y}]^{D_1}. \tag{4.8}$$

Proof. Set $R = \bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma$ and $R' = K[x_2, \ldots, x_n][\mathbf{y}]$. It suffices to show that $\phi_1 : R \to R'$ is an isomorphism. Actually, it implies that $\phi_1(R^D) = (R')^{D_1}$, since $\phi_1 \circ D = D_1 \circ \phi_1$.

First, we show the injectivity of ϕ_1. Suppose that there existed $f \in R \setminus \{0\}$ such that $\phi_1(f) = 0$. Then, $f = (x_1 - 1)f'$ for some $f' \in K[\mathbf{x}][\mathbf{y}] \setminus \{0\}$. We may assume that f' is Γ-homogeneous. Since $x_1 f'$ and f' are Γ-homogeneous components of f, they are in R. However, $\deg_\Gamma(x_1 f') - \deg_\Gamma(f')$ is not in Γ_1. Hence, $\deg_\Gamma(x_1 f')$ or $\deg_\Gamma(f')$ is not in Γ_1. This is a contradiction. Hence, $\phi_1 : R \to R'$ is injective.

For the surjectivity, it suffices to show that $\phi_1(R)$ contains every monomial $\mathbf{x}^a\mathbf{y}^b$ in R'. Put $l=\sum_{j=1}^{n} b_j\epsilon^1_{n,j}$, where $b=(b_1,\ldots,b_n)$. Then, l is nonnegative, since $\epsilon^1_{n,j}\geq 0$ for all j. Hence, $x_1^l\mathbf{x}^a\mathbf{y}^b$ is in $K[\mathbf{x}][\mathbf{y}]$. Take $c\in\mathbf{Z}^m$ such that the image of $(c,\sum_{j=1}^{n} b_j\mathbf{e}_n)$ in Γ is equal to $\deg_\Gamma(x_1^l\mathbf{x}^a\mathbf{y}^b)$. Then, by simple computation, we see that the first component of c is equal to $l+\sum_{j=1}^{n} b_j\epsilon^1_{j,n}=0$. Thus, $x_1^l\mathbf{x}^a\mathbf{y}^b$ is in R. It implies that $\phi_1(R)$ contains $\mathbf{x}^a\mathbf{y}^b$. Therefore, the surjectivity is proved. □

Lemma 4.7 *Assume that $n=4$ and $\epsilon^1_{1,3},\epsilon^1_{1,2}>0,\epsilon^1_{1,4}=0$. Then, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by x_1 and $L_{3,2}$ over $\bigoplus_{\gamma\in\Gamma_1}K[\mathbf{x}][\mathbf{y}]^D_\gamma$.*

Proof. Without loss of generality, we may assume that $\epsilon^1_{1,3}\geq\epsilon^1_{1,2}$. It suffices to show that each Γ-homogeneous element $F\in K[\mathbf{x}][\mathbf{y}]^D$ is written as $F=x_1^pL_{3,2}^qF'$. Here, $p,q\in\mathbf{Z}_{\geq 0}$ and $F'\in K[\mathbf{x}][\mathbf{y}]_{\gamma'}$ for some $\gamma'\in\Gamma_1$. Actually, it also implies that $D(F')=0$, since $0=D(F)=x_1^pL_{3,2}^qD(F')$,

Assume that $\deg_\Gamma(F)$ is equal to the image of $(a,l\mathbf{e}_4)$, where $a=(a_1,\ldots,a_m)\in\mathbf{Z}^m$ and $l\in\mathbf{Z}_{\geq 0}$. We set $f=\phi(F)$. Then, it follows that $F=\tau^l_{\mathbf{x}^a}(f)$, as we noted before Lemma 3.4. Take any $b=(b_1,b_2,b_3,b_4)\in\operatorname{supp}(f)$. Then, by straightforward computation, we get $\tau^l_{\mathbf{x}^a}(\mathbf{y}^b)=\mathbf{x}^c\mathbf{y}^b$, where

$$c=a+(l-b_4)\epsilon_{4,1}+b_2\epsilon_{1,2}+b_3\epsilon_{1,3}. \tag{4.9}$$

Since $\epsilon^1_{4,1}=0$, the first component of c is equal to $a_1+\tilde{l}(b)$. It implies that $\tilde{l}(b)\geq -a_1$. Since $\epsilon^1_{1,2},\epsilon^1_{1,3}>0$, we get $\tilde{l}(b)\geq 0$. So, $x_1^{-a_1}\mathbf{x}^c\mathbf{y}^b$ is in $K[\mathbf{x}][\mathbf{y}]$. Hence, $x_1^{-a_1}F$ is so. Clearly, $\deg_\Gamma(x_1^{-a_1}F)$ is in Γ_1. Therefore, we are led to the desired expression $F=x_1^{a_1}(x_1^{-a_1}F)$ if $a_1\geq 0$.

Assume that $a_1<0$. Let q be the minimal integer such that $q\epsilon^i_{1,3}\geq -a_1$. As mentioned above, every $b\in\operatorname{supp}(f)$ satisfies $\tilde{l}(b)\geq -a_1$. Hence, $(y_3-y_2)^q$ divides f by Lemma 3.5. It implies that $F=F'L_{3,2}^q$ for some $F'\in K[\mathbf{x}][\mathbf{y}]^D$. Note that $\deg_\Gamma(L_{3,2}^q)$ is equal to the image of $q(\epsilon^+_{2,3}+\epsilon_{3,4},\mathbf{e}_4)$ in Γ. Hence, $\deg_\Gamma(F')$ is equal to that of $(a',(l-q)\mathbf{e}_4)$, where

$$a'=a-q(\epsilon^+_{2,3}+\epsilon_{3,4})=a+q\epsilon_{1,3}-q(\epsilon^+_{2,3}+\epsilon_{1,4}). \tag{4.10}$$

Since the first components of $\epsilon_{2,3}^{+}, \epsilon_{1,4}$ are zero, that of a' is equal to $a_1 + q\epsilon_{1,3}^{1}$. By the choice of q, this is nonnegative. Hence, we have $F' = \mathbf{x}^p F''$ for some $p \in \mathbf{Z}_{\geq 0}$ and $F'' \in K[\mathbf{x}][\mathbf{y}]_{\gamma'}$ with $\gamma' \in \Gamma_1$, as we already showed. Therefore, we get a desired expression. This completes the proof. $\square$

Now, let us prove Theorem 4.1. Note that the assumption fails if and only if we can exchange the rows and columns of the matrix $(\delta_i^j)_{i,j}$ so that δ_i^i is the maximum among the components of the i-th column of it for each $i = 1, 2, 3$. So, under the assumption, we are reduced to one of the following two cases by such operations:

(i) $\delta_i^1 \leq \delta_1^1$ and $\delta_i^2 \leq \delta_1^2$ for $i = 1, \ldots, 4$.

(ii) $\delta_i^1 < \delta_1^1 = \delta_4^1$ for $i = 2, 3$.

Actually, if we were not reduced to (ii), then there exists $1 \leq k_j \leq 4$ for each $j = 1, 2, 3$ such that $\delta_i^j < \delta_{k_j}^j$ for any $i \neq k_j$. If further we were not reduced to (i), then $k_j \neq k_l$ for any $j \neq l$. In this case, we can change the rows and columns of $(\delta_i^j)_{i,j}$ so that $k_j = j$ for $j = 1, 2, 3$. This implies that $\delta_i^j < \delta_i^i$ for any $i \neq j$.

First, consider the case (i). By changing the low vectors δ_2, δ_3 and δ_4 of $(\delta_i^j)_{i,j}$ if necessary, we may assume that $\delta_4^3 \leq \delta_j^3$, that is, $\epsilon_{j,4}^3 \geq 0$ for $j = 2, 3, 4$. Furthermore, we have $\epsilon_{1,4}^1, \epsilon_{1,4}^2 \geq 0$, since $\delta_4^1 \leq \delta_1^1$ and $\delta_4^2 \leq \delta_1^2$ by assumption. Hence, $K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][y_1, y_2, y_3]^D[L_{4,1}]$ by Proposition 4.4. By [5, Corollary 2.2], it follows that

$$K[\mathbf{x}][y_1, y_2, y_3]^D = K[\mathbf{x}][L_{2,1}, L_{3,1}, L_{3,2}]. \tag{4.11}$$

Therefore, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by $L_{2,1}$, $L_{3,1}$, $L_{3,2}$ and $L_{4,1}$ over $K[\mathbf{x}]$.

Now, consider the case (ii). Since $\epsilon_{2,1}^1, \epsilon_{3,1}^1 < 0$ and $\epsilon_{4,1}^1 = 0$ follow from the condition, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by $x_1, L_{3,2}^D$ over $\bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma^D$ by Lemma 4.7. Since $\epsilon_{4,j}^1 \geq 0$ for each j, $\bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma^D$ is isomorphic to $K[x_2, x_3][\mathbf{y}]^{D'}$ via ϕ_1 by Lemma 4.6. Then, by Corollary 4.5, there exist $1 \leq l \leq 4$ and $1 \leq k_i \leq 4$ with $k_i \neq i$ for $i \in \{1, \ldots, 4\} \setminus \{l\}$ such that $K[x_2, x_3][\mathbf{y}]^{D'}$ is generated by $L_{k_i,i}^{D'}$ for $i \in \{1, \ldots, 4\} \setminus \{l\}$ over $K[x_2, x_3]$. Since $\phi_1(L_{i,j}^D) = L_{i,j}^{D'}$ for i, j, the $K[x_2, x_3]$-algebra $\bigoplus_{\gamma \in \Gamma_1} K[\mathbf{x}][\mathbf{y}]_\gamma^D$ is generated by $L_{k_i,i}$ for $i \in \{1, \ldots, 4\} \setminus \{l\}$. Therefore, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by $L_{3,2}^D$ and $L_{k_i,i}^D$ for $i \in \{1, \ldots, 4\} \setminus \{l\}$ over $K[\mathbf{x}]$. This completes the proof of Theorem 4.1.

Let D be any elementary monomial $K[\mathbf{x}]$-derivation on $K[\mathbf{x}][\mathbf{y}]$ for $(m,n) = (3,4)$. By Theorems 1.4 and 4.1, we settled the problem of finite generation of $K[\mathbf{x}][\mathbf{y}]^D$ except for the case $\epsilon_{i,j}^i > 0$ for any $i \neq j$ and $\xi(D) > 1$.

Conjecture 4.8 *Assume that* $(m,n) = (3,4)$, *and* $\epsilon_{i,j}^i > 0$ *for any* $i \neq j$. *If* $\xi(D) > 1$, *then* $K[\mathbf{x}][\mathbf{y}]^D$ *is finitely generated.*

Note that the conjecture is true if there exist distinct $r, s \in \{1,2,3\}$ such that $\xi_r(D) \geq 1$ and $\xi_s(D) \geq 1$. We show this for $(r,s) = (2,3)$. The conditions $\xi_2(D) \geq 1$ and $\xi_3(D) \geq 1$ imply, respectively, that $\epsilon_{3,4}^2 \geq 0$ or $\epsilon_{1,4}^2 \geq 0$, and $\epsilon_{1,4}^3 \geq 0$ or $\epsilon_{2,4}^3 \geq 0$. Furthermore, we have $\epsilon_{1,4}^1 > 0$, $\epsilon_{2,4}^2 > 0$ and $\epsilon_{3,4}^3 > 0$ by assumption. Hence, for each $i = 1,2,3$, it follows that $\epsilon_{1,4}^i \geq 0$ or $\epsilon_{l,4}^i \geq 0$ for $l = 2,3,4$. Thus, $K[\mathbf{x}][\mathbf{y}]^D = K[\mathbf{x}][y_2,y_3,y_4]^D[L_{4,1}]$ by Proposition 4.4. By [5, Corollary 2.2] (See (4.11)), $K[\mathbf{x}][y_2,y_3,y_4]^D$ is generated by $L_{3,2}$, $L_{4,2}$ and $L_{4,3}$ over $K[\mathbf{x}]$. Therefore, $K[\mathbf{x}][\mathbf{y}]^D$ is generated by $L_{3,2}$, $L_{4,1}$, $L_{4,2}$ and $L_{4,3}$ over $K[\mathbf{x}]$.

There exists an example of an elementary monomial $K[\mathbf{x}]$-derivation on $K[\mathbf{x}][\mathbf{y}]$ for $(m,n) = (3,4)$ whose kernel is finitely generated, and $\xi_i(D) < 1$ for $i = 1,2,3$. In [7], Kurano showed that the kernel of $D_{1,3}$ is finitely generated. Actually, he showed that it is generated by $L_{i,j}$ for any $(i,j) \in \mathbf{Z} \times \mathbf{Z}$ with $1 \leq j < i \leq 4$ and

$$x_i y_4^2 - 2x_j x_k y_i y_4 + x_i x_k^2 y_i y_j + x_i x_j^2 y_i y_k - x_i^3 y_j y_k \tag{4.12}$$

for $(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$. For this derivation, we have $\xi_i(D_{1,3}) = 1/2$ for $i = 1,2,3$.

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Chapter 4

The infiniteness of the SAGBI bases for certain invariant rings

The infiniteness of the SAGBI bases for certain invariant rings

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Abstract

We consider invariant subrings of a polynomial ring for certain actions of finite groups. We show that, for any multiplicative order, the initial algebra of such an invariant subring is not finitely generated if the finite group is not a direct product of symmetric groups. Furthermore, there exist uncountable cardinality of distinct initial algebras for each invariant subring in this case.

Introduction

The concept of initial ideals for ideals of a polynomial ring in Gröbner basis theory is generalized in a natural way for subalgebras of a polynomial ring, and they are called initial algebras. A set of generators of a subalgebra is called a SAGBI (Subalgebra Analogue to Gröbner Bases for Ideals) basis [6] if their initial monomials generate the initial algebra. The main difference between the initial ideal and the initial algebra is that the former always has finite generators by Hilbert's basis theorem while the latter does not. Hence, it is an important problem to find a criterion for the finite generation of initial algebras.

Göbel [2] studied this problem for the invariant subring $k[\mathbf{x}]^G$ of $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ for a finite group G acting by permutations of the variables. He showed that, with respect to the lexicographic order, the initial algebra of $k[\mathbf{x}]^G$ is finitely generated if and only if G is a direct product of symmetric groups.

In this paper, we prove that a similar result holds for any multiplicative order, i.e. a monomial order which does not require the minimality of the unit.

In case of initial ideals, there exist only finite cardinality of distinct initial ideals for an ideal under a certain condition, although there exist infinite cardinality of orders in general. However, this is not always true in case of initial algebras. Our second result is about the cardinality of distinct initial algebras of invariant rings of permutation groups. We will show that there exist uncountable cardinality of distinct initial algebras for each invariant ring, when G is not a direct product of symmetric groups. If G is a product of symmetric groups, there exist finite cardinality of distinct initial algebras. The exact number is given in Proposition 3.3. We prove similar results on initial algebras for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$, i.e., for invariant subrings of the Laurent polynomial ring $k[\mathbf{x}, \mathbf{x}^{-1}]$.

In Section 1, we introduce a topology on the set of multiplicative orders. This section also contains our notation and the basic definitions. Section 2 presents our main results.

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1 The topological structure of multiplicative orders and standard bases for vector spaces

We fix a field k of an arbitrary characteristic. Let n be a positive integer, and denote by $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ and $k[\mathbf{x}, \mathbf{x}^{-1}] := k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ the polynomial and the Laurent polynomial rings in n variables over k, respectively. Throughout this paper, the monomials in $k[\mathbf{x}, \mathbf{x}^{-1}]$ are denoted $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and identified with lattice points $\mathbf{a} = (a_1, \ldots, a_n)$ in $\mathbf{Z}^n$. An algebra always means a k-algebra.

A total order $\prec$ on $\mathbf{Z}^n$ is said to be *multiplicative* if $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a}+\mathbf{c} \prec \mathbf{b}+\mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{Z}^n$. A *monomial order* is a total order which is a multiplicative order and the zero vector 0 is the minimum element among $\mathbf{Z}^n_{\geq 0}$. We denote by $\mathbf{S}'$ the set of vectors $\omega = (\omega^1, \ldots, \omega^n)$ on the $(n-1)$-dimensional unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ whose components $\omega^1, \ldots, \omega^n \in \mathbf{R}$ are linearly independent over $\mathbf{Q}$. For each $\omega \in \mathbf{S}'$, the multiplicative order $\prec = \iota(\omega)$ is defined by

$$\mathbf{a} \prec \mathbf{b} :\Leftrightarrow \omega \cdot \mathbf{a} \leq \omega \cdot \mathbf{b}.$$

Note that the inner products $\omega \cdot \mathbf{a}$ and $\omega \cdot \mathbf{b}$ are not equal for any distinct $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{Z}^n$ by the linear independence of $\omega^1, \ldots, \omega^n$ over $\mathbf{Q}$.

For a convex polytope $P \subset \mathbf{R}^n$ and $\omega \in \mathbf{R}^n$, the *face* $\mathrm{face}_\omega(P)$ of P is defined by

$$\mathrm{face}_\omega(P) := \{\mathbf{a} \in \mathbf{R}^n \mid \omega \cdot \mathbf{a}' \leq \omega \cdot \mathbf{a} \text{ for all } \mathbf{a}' \in P\}.$$

We denote by Ω the set of multiplicative orders, by Ω_0 the set of monomial orders, and by $\mathcal{V}$ the set of k-vector spaces $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ spanned by monomials.

We introduce topologies on Ω and $\mathcal{V}$ as follows. We take a map ρ from $\mathbf{Z}^n$ to $\mathbf{Z}_{>0}$ such that $\rho^{-1}(l)$ is a finite set for every $l \in \mathbf{Z}_{>0}$. Let $d_\rho : \Omega \times \Omega \to \mathbf{R}$ and $\delta_\rho : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$ be functions defined as follows. For all $\prec, \prec' \in \Omega$, we set

$$d_\rho(\prec, \prec') := \begin{cases} 0 & \text{if } \prec = \prec' \\ 1/e & \text{if } e = \max\{e \in \mathbf{Z}_{>0} \mid \mathbf{x}^\mathbf{a} \prec \mathbf{x}^\mathbf{b} \Leftrightarrow \mathbf{x}^\mathbf{a} \prec' \mathbf{x}^\mathbf{b} \\ & \qquad \text{for all } \mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b} \in k[\mathbf{x}, \mathbf{x}^{-1}] \text{ such that } \rho(\mathbf{a}), \rho(\mathbf{b}) < e\}. \end{cases}$$

For all $V, V' \in \mathcal{V}$, we set

$$\delta_\rho(V, V') := \begin{cases} 0 & \text{if } V = V' \\ 1/e & \text{if } e = \max\{e \in \mathbf{Z}_{>0} \mid \mathbf{x}^\mathbf{a} \in V \Leftrightarrow \mathbf{x}^\mathbf{a} \in V' \\ & \qquad \text{for all } \mathbf{x}^\mathbf{a} \in k[\mathbf{x}, \mathbf{x}^{-1}] \text{ such that } \rho(\mathbf{a}) < e\}. \end{cases}$$

It is easy to see that d_ρ and δ_ρ define metrics of Ω and $\mathcal{V}$, respectively. For $\mathbf{S}'$, we consider the topology induced from $\mathbf{R}^n$.

Theorem 1.1 *The topological structures of the metric spaces* (Ω, d_ρ) *and* $(\mathcal{V}, \delta_\rho)$ *are independent of the choice of* ρ*. The set* Ω *of multiplicative orders is compact with respect to this topology. Furthermore, the injection* $\iota : \mathbf{S}' \to \Omega$ *is continuous. The image* $\iota(\mathbf{S}')$ *is a dense subset of* Ω*.*

Proof. Let d_{ρ_1}, d_{ρ_2} be distance functions on Ω determined by maps ρ_1, ρ_2 from $\mathbf{Z}^n$ to $\mathbf{Z}_{>0}$ as above. We take an arbitrary $\prec \in \Omega$ and $e > 0$. Then, there exists $e' \gg 0$ such that $\{\mathbf{x}^\mathbf{a} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_1(\mathbf{a}) \le e'\}$ and $\{\mathbf{x}^\mathbf{a} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_2(\mathbf{a}) \le e'\}$ contain $\{\mathbf{x}^\mathbf{a} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_1(\mathbf{a}) \le e$ or $\rho_2(\mathbf{a}) \le e\}$. Now, it follows for every $\prec' \in \Omega$ that $d_{\rho_1}(\prec, \prec') < 1/e'$ implies $d_{\rho_2}(\prec, \prec') < 1/e$, and $d_{\rho_2}(\prec, \prec') < 1/e'$ implies $d_{\rho_1}(\prec, \prec') < 1/e$. Hence, d_{ρ_1} and d_{ρ_2} define the same topology.

By a similar argument, we can prove that any two distance functions δ_{ρ_1} and δ_{ρ_2} define the same topology on $\mathcal{V}$.

We prove the totally boundedness of Ω. We take a positive number e. Then the cardinality of monomials $\mathbf{x}^\mathbf{a}$ with $\rho(\mathbf{a}) \le e$ is finite. So, there exist only finite cardinality of distinct orders on the set of monomials $\mathbf{x}^\mathbf{a}$ with $\rho(\mathbf{a}) \le e$. Hence, we can take $\prec_1, \ldots, \prec_l \in \Omega$ such that, for every $\prec \in \Omega$, it follows that $d_\rho(\prec, \prec_i) < 1/e$ for some i. Then the $1/e$-neighborhoods of $\prec_i$'s is a finite $1/e$-covering of Ω.

Now we see the completeness of Ω as follows. Let $\{\prec_i\}_i \subset \Omega$ be a Cauchy sequence. Then, for every integer $e > 0$, there exists an integer $k_e > 0$ such that $d_\rho(\prec_i, \prec_j) < 1/e$ for all $i, j \ge k_e$. Now, $\{\prec_i\}_i$ tends to the order $\prec \in \Omega$ which is defined by

$$\mathbf{x}^\mathbf{a} \prec \mathbf{x}^\mathbf{b} :\Leftrightarrow \mathbf{x}^\mathbf{a} \prec_{k_e} \mathbf{x}^\mathbf{b},$$

where e is an integer greater than $\rho(\mathbf{a})$ and $\rho(\mathbf{b})$.

Finally, we prove the continuity of the injection $\iota : \mathbf{S}' \to \Omega$ and the density of its image. Take any $\omega_0 \in \mathbf{S}'$, and set $\prec_0 = \iota(\omega_0)$. Assume that e is a positive number. Then the following three conditions are equivalent for $\omega \in \mathbf{S}'$ and $\prec = \iota(\omega)$:

(i) $d_\rho(\prec_0, \prec) < 1/e$.

(ii) $\omega_0 \cdot \mathbf{a} \le \omega_0 \cdot \mathbf{b} \Leftrightarrow \omega \cdot \mathbf{a} \le \omega \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a}), \rho(\mathbf{b}) \le e$.

(iii) $\mathrm{face}_\omega(\mathrm{conv}\{\mathbf{a},\mathbf{b}\}) = \mathrm{face}_{\omega_0}(\mathrm{conv}\{\mathbf{a},\mathbf{b}\})$ for all $\mathbf{a},\mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a}), \rho(\mathbf{b}) \le e$.

where $\mathrm{conv}\{\mathbf{a},\mathbf{b}\}$ is the convex hull of $\{\mathbf{a},\mathbf{b}\}$. In general, for a convex polytope $P \subset \mathbf{R}^n$ and a vertex $\{v_0\} = \mathrm{face}_{\eta_0}(P)$, the set $\{\eta \in \mathbf{R}^n \mid \mathrm{face}_\eta(P) = \{v_0\}\}$ of vectors is an open cone of $\mathbf{R}^n$. In particular,

$$U(\mathbf{a},\mathbf{b}) := \{\omega \in \mathbf{S}' \mid \mathrm{face}_\omega(\mathrm{conv}\{\mathbf{a},\mathbf{b}\}) = \mathrm{face}_{\omega_0}(\mathrm{conv}\{\mathbf{a},\mathbf{b}\})\}$$

is an open set of $\mathbf{S}'$. Since $\{\omega \in \mathbf{S}' \mid d_\rho(\prec_0, \iota(\omega)) < 1/e\}$ is the intersection of $U(\mathbf{a},\mathbf{b})$'s for $\mathbf{a},\mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a}), \rho(\mathbf{b}) \le e$, it is an open set of $\mathbf{S}'$. Hence, the map ι is continuous.

The density of $\iota(\mathbf{S}')$ in Ω follows from Robbiano's classification of multiplicative orders [5, Theorem 2.5]:

> *Let $\prec$ be a multiplicative order. Then there exist vectors $\omega_1, \ldots, \omega_N \in \mathbf{R}^n$ such that $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ if and only if $\omega_i \cdot \mathbf{a} < \omega_i \cdot \mathbf{b}$ for the first i such that $\omega_i \cdot \mathbf{a} \neq \omega_i \cdot \mathbf{b}$, for all $\mathbf{a},\mathbf{b} \in \mathbf{Z}^n$.*

Indeed, we set $\omega(T) := \sum_{i=1}^N \omega_i T^{N-i}$ and take $\{t_i\}_i \subset \mathbf{R}$ such that $t_i \to +\infty$ as $i \to +\infty$ and $|\omega(t_i)|^{-1}\omega(t_i) \in \mathbf{S}'$. Then the sequence $\{\iota(|\omega(t_i)|^{-1}\omega(t_i))\}_i$ tends to $\prec$. $\square$

The topology of Ω defined as above is the same as the topology which is defined as follows (cf. [4, Lecture 3], [7]): Let $\Omega \to \{1,-1\}^{\mathbf{Z}^n}$ be the inclusion map which is defined, for each $\prec \in \Omega$, by $\prec(\mathbf{a}) := 1$ if $0 \prec \mathbf{a}$, and -1 otherwise, for all $\mathbf{a} \in \mathbf{Z}^n$. The set $\{1,-1\}^{\mathbf{Z}^n}$ is considered to be the topological space which is the product of the discrete topological space $\{1,-1\}$. The topological structure of Ω is induced from this topology.

In what follows, by a vector space $V \subset k[\mathbf{x},\mathbf{x}^{-1}]$, we mean a vector space over the field k.

Definition 1.2 Let $\prec$ be a multiplicative order, $f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in k[\mathbf{x},\mathbf{x}^{-1}]$ a nonzero polynomial, and $V \subset k[\mathbf{x},\mathbf{x}^{-1}]$ a vector space.

(1) The *initial monomial* of f with respect to $\prec$ is defined by

$$\mathrm{in}_\prec(f) := \max_\prec\{\mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}. \tag{1.1}$$

Then it follows that $\mathrm{in}_\prec(f \cdot g) = \mathrm{in}_\prec(f) \cdot \mathrm{in}_\prec(g)$ for $f, g \in k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$.

(2) The *initial vector space* of V with respect to $\prec$ is by definition the vector space spanned by $\{\mathrm{in}_\prec(f) \mid f \in V \setminus \{0\}\}$. If A is a subalgebra of $k[\mathbf{x}, \mathbf{x}^{-1}]$, then $\mathrm{in}_\prec(A)$ has an algebra structure, since $\mathrm{in}_\prec(f) \cdot \mathrm{in}_\prec(g) = \mathrm{in}_\prec(f \cdot g)$ for any $f, g \in A \setminus \{0\}$. We call it the *initial algebra* of A with respect to $\prec$.

A set S of generators of A is called a *SAGBI basis* with respect to $\prec \in \Omega$, if $\{\mathrm{in}_\prec(f) \mid 0 \neq f \in S\}$ generates $\mathrm{in}_\prec(A)$ as an algebra. Note that A has a finite SAGBI basis only if the initial algebra $\mathrm{in}_\prec(A)$ is finitely generated.

The correspondence $\prec \mapsto \mathrm{in}_\prec(V)$ is a map from the set Ω of multiplicative orders to the set $\mathcal{V}$ of vector spaces spanned by monomials. This map is denoted by F_V. It is not continuous in general. However, if the vector space V satisfies the following separation condition, then F_V is continuous.

> For each monomial m, there exist subspaces $H, K \subset V$ such that $V = H + K$. Here, the number of monomials appearing in polynomials in H is finite, m does not appear in any polynomials in K, and a polynomial in H and a polynomial in K have no common monomials.

Actually, if $F_V(\prec)$ does not contain m, then neither does $F_V(\prec')$ for $\prec'$ in a sufficiently small neighborhood of $\prec$, since $F_V(\prec'') = F_H(\prec'') + F_K(\prec'')$ holds for any $\prec'' \in \Omega$. We denote by $U_V(\prec)$ the inverse image of the initial vector space $\mathrm{in}_\prec(V) \in \mathcal{V}$. Namely,

$$U_V(\prec) := \{\prec' \in \Omega \mid \mathrm{in}_{\prec'}(V) = \mathrm{in}_\prec(V)\}. \tag{1.2}$$

If V satisfies the separation condition, then $U_V(\prec)$ is a closed subset of Ω, because $\mathcal{V}$ is Hausdorff and the map F_V is continuous.

Definition 1.3 Let $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a vector space, and $\prec$ a multiplicative order.

(1) A basis $\{f_i\}_i$ of the vector space V is said to be *standard* with respect to $\prec$, if $\{\mathrm{in}_\prec(f_i)\}_i$ is a basis of the vector space $\mathrm{in}_\prec(V)$.

(2) A polynomial $0 \neq f \in V$ is said to be *reduced*, if all monomials of f but $\mathrm{in}_\prec(f)$ are not contained in $\mathrm{in}_\prec(V)$.

(3) A standard basis $\{f_i\}_i$ is said to be *reduced* if every f_i is reduced.

We remark that the index set of a standard basis of a vector space V with respect to $\prec \in \Omega$ can be taken as the set of monomials in $\mathrm{in}_\prec(V)$. Namely, we denote a standard basis by $\{f_m\}_m$ with $m = \mathrm{in}_\prec(f_m)$ where m runs through the monomials of $\mathrm{in}_\prec(V)$.

The following lemma is well known.

Lemma 1.4 *Let $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a vector space and $\prec, \prec'$ multiplicative orders. Assume that there exists a reduced standard basis of V with respect to $\prec$ and $\prec'$. Then,* $\mathrm{in}_\prec(V) \subset \mathrm{in}_{\prec'}(V)$ *implies* $\mathrm{in}_\prec(V) = \mathrm{in}_{\prec'}(V)$.

Proof. Let $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ be reduced standard bases of V with respect to $\prec$ and $\prec'$ respectively. For each monomial m in $\mathrm{in}_\prec(V)$, it follows that $f'_m = c_m f_m$ for some $c_m \neq 0$. Actually, we choose c_m so that the coefficient of m in $f'_m - c_m f_m$ is zero. Since f_m and f'_m are reduced, none of the monomials of $f'_m - c_m f_m \in V$ lie in $\mathrm{in}_\prec(V)$. Therefore $f'_m - c_m f_m$ is equal to zero. Hence, by replacing f_m with $c_m f_m$, we may assume $f_m = f'_m$ for every monomial m in $\mathrm{in}_\prec(V)$.

Suppose there existed a proper inclusion of $\mathrm{in}_\prec(V)$ to $\mathrm{in}_{\prec'}(V)$. Then, there exists a proper inclusion $\{f_m\}_m \subset \{f'_{m'}\}_{m'}$ of the reduced standard bases. This is a contradiction, since both $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ are bases of V. □

Let $\{f_m\}_m$ and $\{f'_m\}_m$ be reduced standard bases of V with respect to multiplicative orders $\prec$ and $\prec'$, respectively. If $\mathrm{in}_\prec(V) = \mathrm{in}_{\prec'}(V)$ then we have $f'_m = c_m f_m$ for some $c_m \in k \setminus \{0\}$ for each monomial $m \in \mathrm{in}_\prec(V)$, by the proof of Lemma 1.4. Namely, the reduced standard basis of V is uniquely determined by the vector space $\mathrm{in}_\prec(V)$ up to multiplications of elements of $k \setminus \{0\}$. We sometimes say $\{f_m\}_m$ a reduced standard basis with respect to $\mathrm{in}_\prec(V)$.

Lemma 1.5 *Let $V \subset V' \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be vector spaces, and $\prec \in \Omega$. With respect to $\prec$, we suppose that a reduced standard basis of V is a subset of a reduced standard basis of V'. Then it follows that*

$$U_{V'}(\prec) \subset U_V(\prec).$$

Proof. Let $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ be reduced standard bases of V and V' with respect to $\prec$, respectively. Then it follows that

$$U_V(\prec) = \{\prec'' \in \Omega \mid \mathrm{in}_{\prec''}(f_m) = m \text{ for every monomial } m \in \mathrm{in}_{\prec}(V)\}$$

and

$$U_{V'}(\prec) = \{\prec'' \in \Omega \mid \mathrm{in}_{\prec''}(f'_{m'}) = m' \text{ for every monomial } m' \in \mathrm{in}_{\prec}(V')\}.$$

Now we assume that $\{f_m\}_m \subset \{f'_{m'}\}_{m'}$. Then, for each monomial $m \in \mathrm{in}_{\prec}(V)$, $f_m = f'_{m'}$ implies $m = \mathrm{in}_{\prec}(f_m) = \mathrm{in}_{\prec}(f'_{m'}) = m'$. Hence, we have $U_{V'}(\prec) \subset U_V(\prec)$. $\square$

For a vector space $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$, we denote by $\Delta(V)$ the set of multiplicative orders with respect to which reduced standard bases of V exist. Note that $U_V(\prec)$ is contained in $\Delta(V)$ if $\prec \in \Delta(V)$.

Lemma 1.6 *Let $A \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a subalgebra, and $\prec \in \Delta(A)$. If the algebra* $\mathrm{in}_{\prec}(A)$ *is finitely generated, then $U_A(\prec)$ is an open subset of* $\Delta(A)$.

Proof. Let $\{f_m\}_m$ be a reduced standard basis of A with respect to $\mathrm{in}_{\prec}(A)$. For $0 \neq f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we set $\rho(f) := \max\{\rho(\mathbf{a}_i) \mid c_i \neq 0\}$. Then there exists a positive integer e such that $\mathrm{in}_{\prec}(A)$ is generated by its monomials m with $\rho(f_m) \leq e$. We will show that $1/e$-neighborhood of every $\prec' \in U_A(\prec)$ is contained in $U_A(\prec)$. We fix an arbitrary $\prec' \in U_A(\prec)$ and take $\prec'' \in \Delta(A)$ such that $d_\rho(\prec', \prec'') < 1/e$. Note that $\{f_m\}_m$ is a reduced standard basis with respect to $\prec'$ as well. Then monomial $m \in \mathrm{in}_{\prec'}(A)$ is contained in $\mathrm{in}_{\prec''}(A)$ if $\rho(f_m) \leq e$, because $m = \mathrm{in}_{\prec'}(f_m) = \mathrm{in}_{\prec''}(f_m)$ for $\rho(f_m) \leq e$. Since $\mathrm{in}_{\prec}(A) = \mathrm{in}_{\prec'}(A)$ is generated by monomials m with $\rho(f_m) \leq e$, we have $\mathrm{in}_{\prec'}(A) \subset \mathrm{in}_{\prec''}(A)$. This implies $\mathrm{in}_{\prec'}(A) = \mathrm{in}_{\prec''}(A)$ by Lemma 1.4. Hence, $\prec''$ is contained in $U_A(\prec)$. Therefore the $1/e$-neighborhood of $\prec'$ is contained in $U_A(\prec)$. $\square$

The converse of Lemma 1.6 is not true in general. Actually, there exists a subalgebra A of $k[\mathbf{x}, \mathbf{x}^{-1}]$ which is generated by monomials but is not finitely generated. In this case, $U_A(\prec) = \Delta(A) = \Omega$ for any $\prec \in \Omega$.

Let I be an ideal of $k[\mathbf{x}]$. By Hilbert's basis theorem, the ideal $\mathrm{in}_{\prec}(I)$ is always finitely generated. By the argument similar to Lemma 1.6, Schwartz [7, Theorems 13 and 30] showed that, for any subset G of I,

$$U_{I,G} := \{\prec \in \Omega_0 \mid G \text{ is a Gröbner basis of } I \text{ with respect to } \prec\} \tag{1.3}$$

is an open subset of Ω_0. Note that Ω_0 is a compact subset of Ω. In fact, we have the following lemma.

Lemma 1.7 *Let $S \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be an algebra which is generated by a finite subset of monomials in S. Then the set of multiplicative orders which are well-orderings on the set of monomials in S is compact (may be empty).*

Proof. We remark that $\prec \in \Omega$ is a well-ordering on the set of monomials in S, if and only if the unit 1 is the minimum element among the monomials in S. Indeed, if there exists a monomial $1 \neq \mathbf{x}^{\mathbf{a}} \in S$ with $\mathbf{x}^{\mathbf{a}} \prec 1$, then $\{\mathbf{x}^{l\mathbf{a}} \mid l = 1, 2, \ldots\} \subset S$ does not have the minimum element. For the converse, suppose that every monomial of S is greater than 1. Since S is Noetherian, the ideal $(U) \subset S$ is finitely generated (say, by $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\} \subset U$) for any subset U of monomials in S. Then we have $\min_{\prec} U = \min_{\prec}\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\}$.

We set W the set of multiplicative orders which are not well-orderings on the set of monomials in S. We will show that W is an open subset of Ω. For $\prec \in W$, there exists a monomial $1 \neq \mathbf{x}^{\mathbf{a}} \in S$ with $\mathbf{x}^{\mathbf{a}} \prec 1$. We take a positive number e which is greater than $\rho(0)$ and $\rho(\mathbf{a})$. For any multiplicative order $\prec'$ in the $1/e$-neighborhood of $\prec$, we have $\mathbf{x}^{\mathbf{a}} \prec' 1$. So $\prec'$ is not a well-ordering on the set of monomials in S as well. Hence, the $1/e$-neighborhood of $\prec$ is contained in W. Therefore W is open. □

By using the compactness of Ω_0, Schwartz [7, Corollaries 16 and 31] showed the finiteness of the cardinality of distinct initial ideals for a fixed ideal of $k[\mathbf{x}]$ with respect to monomial orders. By the similar argument, we get the following proposition.

Proposition 1.8 *Let $A \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a subalgebra, and Δ a compact subset of $\Delta(A)$. Assume that the initial algebras $\mathrm{in}_{\prec}(A)$ are finitely generated for all $\prec \in \Delta$. Then there exist only finite distinct $\mathrm{in}_{\prec}(A)$'s when $\prec$ runs over Δ.*

Proof. By Lemma 1.6, $U_A(\prec)$ is an open subset of $\Delta(A)$ for any $\prec \in \Delta$. Hence,

$$\{U_A(\prec) \cap \Delta \mid \prec \in \Delta\}$$

is a disjoint open covering of Δ. Since Δ is compact, it is a finite covering. Therefore, the cardinality of distinct initial algebras for A with respect to $\prec \in \Delta$ is finite. $\square$

2 Main result

Throughout Sections 2 and 3, we fix a subgroup G of the symmetric group S_n of degree n. The action of G on $k[\mathbf{x}, \mathbf{x}^{-1}]$ is defined by $\sigma(f) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for $\sigma \in G$ and $f = f(x_1, \ldots, x_n) \in k[\mathbf{x}, \mathbf{x}^{-1}]$. Let $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ and $k[\mathbf{x}]^G$ be the invariant subrings of $k[\mathbf{x}, \mathbf{x}^{-1}]$ and $k[\mathbf{x}]$, respectively, by the action of G.

Recall the following result by Göbel.

Theorem 2.1 (Göbel [2]) *Let $\prec_{\text{lex}} \in \Omega$ be a lexicographic order. Then $\text{in}_{\prec_{\text{lex}}}(k[\mathbf{x}]^G)$ is finitely generated if and only if G is a direct product of symmetric groups.*

Here, by symmetric groups, we mean those of subsets of $\{1, \ldots, n\}$. Note that G is a direct product of symmetric groups if and only if G is generated by the set of transpositions in G. We will show that similar results hold for any multiplicative orders.

Theorem 2.2 *Assume that G is not a direct product of symmetric groups. Then the initial algebra $\text{in}_{\prec}(k[\mathbf{x}]^G)$ is not finitely generated for any multiplicative order $\prec \in \Omega$. There are uncountable cardinality of distinct initial algebras for $k[\mathbf{x}]^G$.*

We get a similar result for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ as follows.

Theorem 2.3 *Assume that G is not a direct product of symmetric groups. Then the initial algebra $\text{in}_{\prec}(k[\mathbf{x}, \mathbf{x}^{-1}]^G)$ is not finitely generated for any multiplicative order $\prec \in \Omega$. There are uncountable cardinality of distinct initial algebras for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$.*

For a subgroup G of a symmetric group and a monomial $\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we define

$$f_G(\mathbf{x}^{\mathbf{a}}) := \sum_{\sigma \in G/G(\mathbf{x}^{\mathbf{a}})} \sigma(\mathbf{x}^{\mathbf{a}}), \tag{2.1}$$

where $G(\mathbf{x}^{\mathbf{a}})$ is the stabilizer $\{\tau \in G \mid \tau(\mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathbf{a}}\}$. We set

$$B := \{f_G(\mathbf{x}^{\mathbf{a}}) \mid \mathbf{a} \in \mathbf{Z}^n\} \tag{2.2}$$

and

$$B_0 := \{f_G(\mathbf{x}^{\mathbf{a}}) \mid \mathbf{a} \in \mathbf{Z}^n_{\geq 0}\}. \tag{2.3}$$

Lemma 2.4 *For any multiplicative order, the sets B and B_0 are reduced standard bases of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ and $k[\mathbf{x}]^G$, respectively.*

Proof. We fix an arbitrary multiplicative order $\prec$. We first remark that if $f_G(\mathbf{x}^{\mathbf{a}})$ and $f_G(\mathbf{x}^{\mathbf{b}})$ have common terms then $f_G(\mathbf{x}^{\mathbf{a}}) = f_G(\mathbf{x}^{\mathbf{b}})$. This implies that B is linearly independent over k, and every $f_G(\mathbf{x}^{\mathbf{a}}) \in B$ is reduced.

We show that B spans $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ over k. Assume that $f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i}$ is an element in $k[\mathbf{x}, \mathbf{x}^{-1}]^G \setminus \{0\}$. Then, G acts on $\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}$. We decompose it into orbits as

$$\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\} = \coprod_l \{c_{i_l} \sigma(\mathbf{x}^{\mathbf{a}_{i_l}}) \mid \sigma \in G\}.$$

The sum of the elements of $\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}$ is equal to f, and the sum of the elements of $\{\sigma(\mathbf{x}^{\mathbf{a}_{i_l}}) \mid \sigma \in G\}$ is equal to $f_G(\mathbf{x}^{\mathbf{a}_{i_l}})$. Hence, we have

$$f = \sum_l c_{i_l} f_G(\mathbf{x}^{\mathbf{a}_{i_l}}).$$

Now, we show that B is a standard basis of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ with respect to $\prec$. Since B spans $k[\mathbf{x}, \mathbf{x}^{-1}]^G$, a G-invariant of $k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$ has an expression $f = \sum_i c_i f_G(\mathbf{x}^{\mathbf{a}_i})$. By the remark, the monomial $\mathrm{in}_\prec(f_G(\mathbf{x}^{\mathbf{a}_i}))$ appears in f with nonzero coefficient if $c_i \neq 0$. Hence, we have

$$\mathrm{in}_\prec(f) = \max{}_\prec\{\mathrm{in}_\prec(f_G(\mathbf{x}^{\mathbf{a}_i})) \mid c_i \neq 0\} \in \{\mathrm{in}_\prec(g) \mid g \in B\}. \tag{2.4}$$

Thus, B is a standard basis of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ with respect to $\prec$.

We show that B_0 is a standard basis of $k[\mathbf{x}]^G$ with respect to $\prec$. Assume that $f = \sum_i c_i f_G(\mathbf{x}^{\mathbf{a}_i})$ is in $k[\mathbf{x}]^G \setminus \{0\}$. By the remark, any term which appears in $c_i f_G(\mathbf{x}^{\mathbf{a}_i})$ appears in f as well. So, each $c_i f_G(\mathbf{x}^{\mathbf{a}_i})$ is an element of $k[\mathbf{x}]$. Hence, B_0 spans $k[\mathbf{x}]^G$. As (2.4), we have $\mathrm{in}_\prec(f) \in \{\mathrm{in}_\prec(g) \mid g \in B_0\}$. Thus B_0 is a standard basis of $k[\mathbf{x}]^G$. $\square$

By this lemma, we have

$$\Delta(k[\mathbf{x}]^G) = \Delta(k[\mathbf{x}, \mathbf{x}^{-1}]^G) = \Omega.$$

Furthermore, it is easy to see that $k[\mathbf{x}]^G$ and $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ satisfy the separation condition which we explained after Definition 1.2. Hence, $U_{k[\mathbf{x}]^G}(\prec)$ and $U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec)$ are closed for any $\prec \in \Omega$.

The following is the key lemma.

Lemma 2.5 *Assume that G is not a direct product of symmetric groups. Then every $\omega \in \mathbf{S}'$ is not an interior point of*

$$\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec)) = \{\omega' \in \mathbf{S}' \mid \mathrm{in}_\prec(k[\mathbf{x}]^G) = \mathrm{in}_{\prec'}(k[\mathbf{x}]^G) \ for \prec' = \iota(\omega')\}$$

for $\prec = \iota(\omega)$, with respect to the Euclidean topology.

Before we prove this lemma, we will prove Theorems 2.2 and 2.3 by assuming this lemma. Let $\prec$ be a multiplicative order. Suppose $\mathrm{in}_\prec(k[\mathbf{x}]^G)$ was finitely generated. Then by Lemma 1.6, $U_{k[\mathbf{x}]^G}(\prec)$ is a nonempty open subset of Ω. The inverse image $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$ is a nonempty open subset of $\mathbf{S}'$ by Theorem 1.1. For $\omega' \in \iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$, we set $\prec' = \iota(\omega')$. Then it follows that $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec')) = \iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$, which implies that ω' is an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec'))$. This contradicts Lemma 2.5. Therefore $\mathrm{in}_\prec(k[\mathbf{x}]^G)$ is not finitely generated.

The set $U_{k[\mathbf{x}]^G}(\prec)$ can not contain interior points by Lemma 2.5, and also it is closed. Hence, it is a nowhere dense subset of Ω. Suppose that there were only countable cardinality of distinct initial algebras for $k[\mathbf{x}]^G$. Then Ω is covered by countable cardinality of $U_{k[\mathbf{x}]^G}(\prec)$'s. Since Ω is a compact metric space, this contradicts the Baire theorem which says that the complement of the union of countable cardinality of nowhere dense subsets of a complete metric space is dense.

By Lemma 2.4, we see that a reduced standard basis of $k[\mathbf{x}]^G$ is a subset of that of $k[\mathbf{x},\mathbf{x}^{-1}]^G$. Hence, we have

$$U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec) \subset U_{k[\mathbf{x}]^G}(\prec)$$

by Lemma 1.5. Since $U_{k[\mathbf{x}]^G}(\prec)$ is nowhere dense, the subset $U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec)$ is also nowhere dense and is not open. Hence, $\mathrm{in}_{\prec}(k[\mathbf{x},\mathbf{x}^{-1}]^G)$ is not finitely generated by Lemma 1.6. The disjoint covering $\{U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec) \mid \prec \in \Omega\}$ of Ω is a refinement of $\{U_{k[\mathbf{x}]^G}(\prec) \mid \prec \in \Omega\}$. Hence, the cardinality of $\mathrm{in}_{\prec}(k[\mathbf{x},\mathbf{x}^{-1}]^G)$'s is uncountable.

The rest of this section is devoted to the proof of Lemma 2.5. Our strategy is to translate polynomial informations into the geometry of convex polytopes. Let

$$\overline{\mathcal{M}} := \left\{(a_1,\ldots,a_n) \in \mathbf{R}^n_{\geq 0} \mid \sum_{i=1}^n a_i = 1\right\} \tag{2.5}$$

and $\mathcal{M} := \overline{\mathcal{M}} \cap \mathbf{Q}^n$. We define the surjection

$$\pi : \left\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbf{Z}^n_{\geq 0} \setminus \{0\}\right\} \to \mathcal{M} \tag{2.6}$$

by $\mathbf{x}^{\mathbf{a}} \mapsto (\sum_{i=1}^n a_i)^{-1}\mathbf{a}$ for $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbf{Z}^n_{\geq 0}$. The action of G on $\overline{\mathcal{M}}$ is by definition $\sigma(\mathbf{a}) := (a_{\sigma(1)},\ldots,a_{\sigma(n)})$ for $\mathbf{a} = (a_1,\ldots,a_n) \in \overline{\mathcal{M}}$ and $\sigma \in G$. For each point $\mathbf{a} \in \overline{\mathcal{M}}$, we denote by $P_G(\mathbf{a})$ the convex hull of the G-orbit $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Note that the set of vertices of $P_G(\mathbf{a})$ is $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$, for each point in $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$ lies on the sphere $\{\mathbf{a}' \in \overline{\mathcal{M}} \mid |\mathbf{a}'| = |\mathbf{a}|\}$.

Let $\prec$ be a multiplicative order defined by $\omega \in \mathbf{S}'$. Then, for each element $\mathbf{a} \in \mathbf{Z}^n_{\geq 0}$, we have $\mathrm{face}_\omega(P_G(\pi(\mathbf{a}))) = \{\pi(\mathbf{a})\}$ if and only if $\mathrm{in}_{\prec}(f_G(\mathbf{x}^{\mathbf{a}})) = \mathbf{x}^{\mathbf{a}}$. By Lemma 2.4, we get the following lemma.

Lemma 2.6 *Assume that* $\prec \in \Omega$ *is defined by* $\omega \in \mathbf{S}'$. *Then*

$$\bigcup_{\mathbf{a}\in\mathcal{M}} \pi^{-1}(\mathrm{face}_\omega(P_G(\mathbf{a}))) \cup \{1\}$$

is a basis of the vector space $\mathrm{in}_{\prec}(k[\mathbf{x}]^G)$. *For* $\omega, \omega' \in \mathbf{S}'$, *set* $\prec = \iota(\omega)$ *and* $\prec' = \iota(\omega')$. *If there exists* $\mathbf{a} \in \mathcal{M}$ *with* $\mathrm{face}_\omega(P_G(\mathbf{a})) \neq \mathrm{face}_{\omega'}(P_G(\mathbf{a}))$, *then* $\mathrm{in}_{\prec}(k[\mathbf{x}]^G) \neq \mathrm{in}_{\prec'}(k[\mathbf{x}]^G)$.

Figures 1 and 2 show the examples of $P_G(\mathbf{a})$'s for $n = 3$. Figure 1 is for $G = S_3$ and Figure 2 is for $G = A_3$.

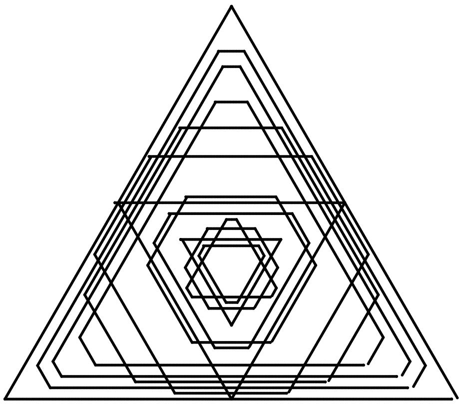

Figure 1: Some $P_{S_3}(\mathbf{a})$'s in $\overline{\mathcal{M}}$.

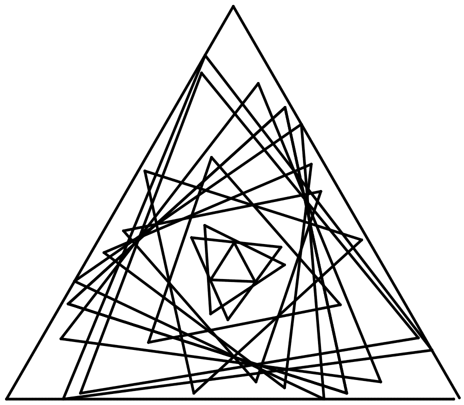

Figure 2: Some $P_{A_3}(\mathbf{a})$'s in $\overline{\mathcal{M}}$.

We will construct a "deformation" of a polytope $P_G(\mathbf{a})$, when G is not a direct product of symmetric groups.

We set $I_\sigma := \{\mathbf{a} \in \overline{\mathcal{M}} \mid \sigma(\mathbf{a}) = \mathbf{a}\}$ for each $\sigma \in G$, and let I be the union of I_σ's for $\sigma \in G \setminus \{1\}$. Then $\overline{\mathcal{M}} \setminus I$ consists of finite number of connected components. For $1 \neq \sigma \in G$ the condition that I_σ has codimension one is equivalent to that σ is a transposition. Since $\overline{\mathcal{M}}$ is a convex set of dimension $n-1$, it is connected even if we remove finite number of linear subspaces of codimension greater than one from it.

Lemma 2.7 *Assume that G is not a direct product of symmetric groups. Then for all $\mathbf{a} \in \overline{\mathcal{M}} \setminus I$, every connected component of $\overline{\mathcal{M}} \setminus I$ contains at least two points of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$.*

Proof. Let $\tau \in G$ be a transposition. Then, the action of τ is the reflection of $\overline{\mathcal{M}}$ with respect to the hyperplane I_τ. For every $1 \neq \sigma \in G$, the subset I_σ of $\overline{\mathcal{M}}$ is the reflection of $I_{\tau\sigma\tau}$ in the hyperplane I_τ. So, the union I of them is symmetric with respect to I_τ. The complement $\overline{\mathcal{M}} \setminus I$ is also symmetric with respect to I_τ.

Now, let $C \neq C'$ be connected components of $\overline{\mathcal{M}} \setminus I$. We will show that $C' = \tau_l \circ \cdots \circ \tau_1(C)$ for some transpositions $\tau_1, \ldots, \tau_l \in G$. Let $\phi : [0,1] \to \overline{\mathcal{M}}$ be a path from a point in C to a point in C'. We assume that ϕ does not intersect $I_\tau \cap I_{\tau'}$ for any transpositions $\tau \neq \tau'$ in S_n, and

$$\{t \in [0,1] \mid \phi(t) \in I_\tau \text{ for some transposition } \tau \in G\}$$

is a finite set, say $\{t_1, \ldots, t_l\}$ with $t_i < t_{i+1}$. We set τ_i the transposition in G with $\phi(t_i) \in I_{\tau_i}$. Then we have $C' = \tau_l \circ \cdots \circ \tau_1(C)$.

We remark that every connected component contains the same cardinality of points of $\{\sigma(\mathbf{a}') \mid \sigma \in G\}$ for each $\mathbf{a}' \in \overline{\mathcal{M}}$. Suppose that there existed a point $\mathbf{a} \in \overline{\mathcal{M}} \setminus I$ and a connected component of $\overline{\mathcal{M}} \setminus I$ which contains only one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Then every connected component contains only one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Assume that $\mathbf{a}$ is contained in a connected component C of $\overline{\mathcal{M}} \setminus I$. For each $1 \neq \sigma \in G$, we have $\sigma(\mathbf{a}) \neq \mathbf{a}$ because $\mathbf{a}$ is not an element of I. Hence, there exists a connected component $C' \neq C$ of $\overline{\mathcal{M}} \setminus I$ such that $\sigma(\mathbf{a}) \in C'$. If $\tau_1, \ldots, \tau_l \in G$ are transpositions such that $C' = \tau_l \circ \cdots \circ \tau_1(C)$, then $\sigma(\mathbf{a}) = \tau_l \circ \cdots \circ \tau_1(\mathbf{a})$ since C' contains exactly one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Because $\mathbf{a}$ is not fixed by any element of $G \setminus \{1\}$, we see that $\sigma = \tau_l \circ \cdots \circ \tau_1$. Therefore G can be generated by transpositions in G. This contradicts the assumption. $\square$

The proof of Lemma 2.5. We fix an arbitrary $\omega \in \mathbf{S}'$ and set $\prec = \iota(\omega)$. We will prove that ω is not an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$.

Let $\mathbf{a} \in \mathcal{M} \setminus I$ such that $\{\mathbf{a}\} = \mathrm{face}_\omega(\mathbf{a})$. Then, by Lemma 2.7, there exists another point $\sigma(\mathbf{a}) \neq \mathbf{a}$, for some $\sigma \in G$, in the connected component of $\overline{\mathcal{M}} \setminus I$ which contains $\mathbf{a}$. We define a path $\gamma : [0,1] \to \overline{\mathcal{M}} \setminus I$ with $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \sigma(\mathbf{a})$ by combining rational points of $\mathcal{M} \setminus I$ with line segments. Then $\gamma([a,b])$ contains rational points densely for any $0 \leq a < b \leq 1$. Now,

$$T = \{t \in [0,1] \mid \omega \cdot \gamma(t) = \omega \cdot \sigma'(\gamma(t)) \ \text{ for some } \sigma' \in G\}$$

is not an empty set. Indeed, since

$$\omega \cdot (\gamma(0) - \sigma^{-1}(\gamma(0))) = \omega \cdot (\mathbf{a} - \sigma^{-1}(\mathbf{a})) > 0$$

and

$$\omega \cdot (\gamma(1) - \sigma^{-1}(\gamma(1))) = \omega \cdot (\sigma(\mathbf{a}) - \mathbf{a}) < 0,$$

there exists $t \in (0,1)$ such that $\omega \cdot (\gamma(t) - \sigma^{-1}(\gamma(t))) = 0$ by the intermediate value theorem. We set $t_0 := \inf(T)$, and $\mathbf{b} := \gamma(t_0)$. Then we have

$$\omega \cdot \mathbf{b} = \omega \cdot \sigma_0(\mathbf{b})$$

for some $1 \neq \sigma_0 \in G$, and

$$\omega \cdot \gamma(t) > \omega \cdot \sigma'(\gamma(t)) \tag{2.7}$$

for all $t \in [0, t_0)$ and $1 \neq \sigma' \in G$. Note that $\mathbf{b} \neq \sigma_0(\mathbf{b})$, since the path γ does not intersect I. For each $\delta \in \mathbf{R}_{>0}$, we set

$$\omega_\delta := \omega - \delta(\mathbf{b} - \sigma_0(\mathbf{b})).$$

Let $\{t_i\}_i \subset [0, t_0)$ be a sequence such that $\lim_{i\to\infty} t_i = t_0$ and $\mathbf{a}_i := \gamma(t_i) \in \mathcal{M}$. Then, for each $\varepsilon' > 0$, there exists a positive integer $N_{\varepsilon'}$ such that

$$|(\mathbf{b} - \sigma_0(\mathbf{b})) \cdot ((\mathbf{b} - \sigma_0(\mathbf{b})) - (\mathbf{a}_i - \sigma_0(\mathbf{a}_i)))| < \varepsilon'$$

and

$$0 < \omega \cdot (\mathbf{a}_i - \sigma_0(\mathbf{a}_i)) < \varepsilon'$$

for every integer $i > N_{\varepsilon'}$.

Now, let ε be any positive number. Then there exists $\delta > 0$ such that

$$\left| \omega - \frac{\omega_\delta}{|\omega_\delta|} \right| < \varepsilon$$

and $|\omega_\delta|^{-1}\omega_\delta \in \mathbf{S}'$. We set $\varepsilon' = (1+\delta)^{-1}\delta|\mathbf{b} - \sigma_0(\mathbf{b})|^2$. Then, for any integer $i > N_{\varepsilon'}$, we have

$$\begin{aligned}
\omega_\delta \cdot (\sigma_0(\mathbf{a}_i) - \mathbf{a}_i) &= (\omega - \delta(\mathbf{b} - \sigma_0(\mathbf{b}))) \cdot (\sigma_0(\mathbf{a}_i) - \mathbf{a}_i) \\
&= \omega \cdot (\sigma_0(\mathbf{a}_i) - \mathbf{a}_i) \\
&\quad -\delta(\mathbf{b} - \sigma_0(\mathbf{b})) \cdot \{((\mathbf{b} - \sigma_0(\mathbf{b})) - (\mathbf{a}_i - \sigma_0(\mathbf{a}_i))) - (\mathbf{b} - \sigma_0(\mathbf{b}))\} \\
&> -\varepsilon' - \delta\varepsilon' + \delta|\mathbf{b} - \sigma_0(\mathbf{b})|^2 = 0.
\end{aligned}$$

Hence,

$$\mathrm{face}_{\omega_\delta}(P_G(\mathbf{a}_i)) \neq \{\mathbf{a}_i\}$$

for $i > N_{\varepsilon'}$. On the other hand, $\{\mathbf{a}_i\} = \mathrm{face}_\omega(P_G(\mathbf{a}_i))$ for all i by (2.7). So, we have $\mathrm{in}_{\prec_\delta}(k[\mathbf{x}]^G) \neq \mathrm{in}_\prec(k[\mathbf{x}]^G)$ for $\prec_\delta = \iota(|\omega_\delta|^{-1}\omega_\delta)$ by Lemma 2.6. Thus, $|\omega_\delta|^{-1}\omega_\delta$ is not in $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$. Therefore ω is not an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$. $\square$

3 Finite SAGBI bases

Now we will observe the case where G is a direct product of symmetric groups.

Lemma 3.1 *Let A be $k[\mathbf{x}]^{S_n}$ or $k[\mathbf{x},\mathbf{x}^{-1}]^{S_n}$. We consider the initial algebras* $\mathrm{in}_\prec(A)$ *for all multiplicative orders $\prec$. Then the cardinality of distinct initial algebras for A is $n!$.*

Proof. It suffices to show that, if $\prec$ and $\prec'$ are multiplicative orders with $x_n \prec \cdots \prec x_1$ and $x_n \prec' \cdots \prec' x_1$, then $\mathrm{in}_\prec(A) = \mathrm{in}_{\prec'}(A)$. By Lemma 2.4, we see that a reduced standard basis of A is equal to

$$\begin{cases} \{f_{S_n}(x_1^{a_1}\cdots x_n^{a_n}) \mid 0 \le a_n \le \cdots \le a_1\} & \text{if } A = k[\mathbf{x}]^{S_n} \\ \{f_{S_n}(x_1^{a_1}\cdots x_n^{a_n}) \mid a_n \le \cdots \le a_1\} & \text{if } A = k[\mathbf{x},\mathbf{x}^{-1}]^{S_n}. \end{cases}$$

For every $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbf{Z}^n$ with $a_n \le \cdots \le a_1$, it follows that $\mathrm{in}_\prec(f_{S_n}(\mathbf{x}^\mathbf{a})) = \mathrm{in}_{\prec'}(f_{S_n}(\mathbf{x}^\mathbf{a})) = \mathbf{x}^\mathbf{a}$. This implies that $\mathrm{in}_\prec(A) = \mathrm{in}_{\prec'}(A)$. $\square$

By the proof of Lemma 3.1, the initial algebras $\mathrm{in}_\prec(k[\mathbf{x}]^{S_n})$ and $\mathrm{in}_\prec(k[\mathbf{x},\mathbf{x}^{-1}]^{S_n})$ are spanned by the sets of monomials

$$\{x_1^{a_1}\cdots x_n^{a_n} \mid 0 \le a_n \le \cdots \le a_1\} \text{ and } \{x_1^{a_1}\cdots x_n^{a_n} \mid a_n \le \cdots \le a_1\},$$

respectively, if the multiplicative order $\prec$ satisfies $x_n \prec \cdots \prec x_1$. In this case, they are generated as algebras by

$$\{x_1, x_1x_2, \ldots, x_1x_2\cdots x_n\} \text{ and } \{x_1, x_1x_2, \ldots, x_1x_2\cdots x_n, x_1^{-1}x_2^{-1}\cdots x_n^{-1}\},$$

respectively. Therefore, the initial algebras $\mathrm{in}_\prec(k[\mathbf{x}]^{S_n})$ and $\mathrm{in}_\prec(k[\mathbf{x},\mathbf{x}^{-1}]^{S_n})$ are finitely generated for any multiplicative order $\prec$ (cf. Robbiano, Sweedler [6, Theorem 1.14]).

Lemma 3.2 (cf. [2, Lemma 3.8]) *Let G_1 and G_2 be subgroups of S_n which acts on $\mathbf{x}_1 := (x_1, \dots, x_l)$ and $\mathbf{x}_2 := (x_{l+1}, \dots, x_n)$, respectively. We set $G = G_1 \times G_2$ the direct product of G_1 and G_2. If A is $k[\mathbf{x}]^G$ or $k[\mathbf{x}, \mathbf{x}^{-1}]^G$, and A_i is $k[\mathbf{x}_i]^{G_i}$ or $k[\mathbf{x}_i, \mathbf{x}_i^{-1}]^{G_i}$ for $i = 1, 2$, respectively, then we have*

$$\mathrm{in}_\prec(A) = \mathrm{in}_\prec(A_1) \otimes_k \mathrm{in}_\prec(A_2).$$

Proof. By Lemma 2.4, the assertion follows from the equality

$$\begin{aligned} f_G(\mathbf{x}_1^{\mathbf{a}_1} \cdot \mathbf{x}_2^{\mathbf{a}_2}) &= \sum_{(\sigma_1,\sigma_2) \in G_1/G_1(\mathbf{x}_1^{\mathbf{a}_1}) \times G_2/G_2(\mathbf{x}_2^{\mathbf{a}_2})} \sigma_1(\mathbf{x}_1^{\mathbf{a}_1}) \cdot \sigma_2(\mathbf{x}_2^{\mathbf{a}_2}) \\ &= \left(\sum_{\sigma_1 \in G_1/G_1(\mathbf{x}_1^{\mathbf{a}_1})} \sigma_1(\mathbf{x}_1^{\mathbf{a}_1}) \right) \cdot \left(\sum_{\sigma_2 \in G_2/G_2(\mathbf{x}_2^{\mathbf{a}_2})} \sigma_2(\mathbf{x}_2^{\mathbf{a}_2}) \right) \\ &= f_{G_1}(\mathbf{x}_1^{\mathbf{a}_1}) \cdot f_{G_2}(\mathbf{x}_2^{\mathbf{a}_2}) \end{aligned}$$

for every monomial $\mathbf{x}_1^{\mathbf{a}_1} \in k[\mathbf{x}_1, \mathbf{x}_1^{-1}]$ and $\mathbf{x}_2^{\mathbf{a}_2} \in k[\mathbf{x}_2, \mathbf{x}_2^{-1}]$. □

Proposition 3.3 *Let A be $k[\mathbf{x}]^G$ or $k[\mathbf{x}, \mathbf{x}^{-1}]^G$. Assume that G is a direct product of symmetric groups. Then the initial algebra $\mathrm{in}_\prec(A)$ is finitely generated for any multiplicative order $\prec$. The cardinality of distinct initial algebras for A is $|G|$.*

Proof. Assume that $n = n_1 + \cdots + n_r$ and $G = S_{n_1} \times \cdots \times S_{n_r}$, and that S_{n_i} acts on the set of variables $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i})$ for each i. Let A_i be $k[\mathbf{x}_i]^{S_{n_i}}$ if $A = k[\mathbf{x}]^G$, and $k[\mathbf{x}_i, \mathbf{x}_i^{-1}]^{S_{n_i}}$ if $A = k[\mathbf{x}, \mathbf{x}^{-1}]^G$. Then there exist $n_i!$ distinct initial algebras for each A_i by Lemma 3.1. Since we can define the order in $\mathbf{x}_i$ independently for each i, there exist $n_1! \cdots n_r!$ distinct initial algebras for A. Clearly, this number is equal to the order of the group G.

Since each A_i is finitely generated for any $\prec \in \Omega$, the tensor product of them is also finitely generated. Hence, the initial algebra $\mathrm{in}_\prec(A)$ is finitely generated for any $\prec \in \Omega$. □

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